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Algorithmic models of market equilibrium

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Abstract

In this paper we suggest a new framework for constructing mathematical models of market activity. Contrary to the majority of the classical economical models (e.g. Arrow-Debreu, Walras, etc.), we get a characterization of general equilibrium of the market as a saddle point in a convex-concave game. This feature significantly simplifies the proof of existence theorems and construction of the adjustment processes both for producers and consumers. Moreover, we argue that the unique equilibrium prices can be characterized as a unique limiting point of some simple price dynamics. In our model, the equilibrium prices have natural explanation: they minimize the total excessive revenue of the market's participants.

Due to convexity, all our adjustment processes have unambiguous behavioral and algorithmic interpretation. From the technical point of view, the most unusual feature of our approach is the absence of the budget constraint in its classical form.

Keywords: general equilibrium theory, convex optimization, price mechanism, budget constraint

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1 Introduction

In the microeconomic theory, the concept of equilibrium is may be the most fundamental. It equalizes the volumes of production and consumption of goods by appropriate equilibrium prices. This concept arises in the framework of *neoclassical economics*, originated from the pioneering papers by Cournot [5], Walras [17], and Marshal [11]. The general equilibrium theory was finalized in the seminal paper by Arrow and Debreu [4], where the authors proved the existence of equilibrium prices under very mild and natural assumptions. Earlier in [2], Arrow has already shown the validity of welfare theorems roughly stating that competitive markets tend toward an equilibrium allocation of resources which is efficient from a social point of view. The subsequent developments were related mainly with polishing some elements of this beautiful theory [12], [13].

Recall that in the *Arrow-Debreu model*, each consumer makes his/her choice in the *space of goods* by *maximizing* the utility function subject to the *budget constraint*. This framework perfectly fits our intuition. However, it can be criticized from the viewpoint of computability and implementability.

Indeed, in the last decades we have learned from Optimization Theory that the general optimization problems are may be the most difficult problems in Numerical Analysis. They are very difficult (and will be always difficult) even for the most powerful computers. How comes that a usual household, who has natural weakness even in the trivial arithmetics, is able to solve in mind the optimization problems with tens thousands variables? (This is the usual number of goods in the modern supermarkets.) For that, the standard textbooks on microeconomics ([9]) advise to compute the gradient of the utility function by estimating the marginal utilities of all goods. This procedure may fail by the following reasons. First of all, in order to compute the marginal utility of one good, we need to buy and try it. Clearly, in view of the high dimension, this cannot be implemented in the real life. Secondly, this computation is useful only for estimating the gradients of *differentiable* functions. However, the differentiability assumption on utility functions seems to be rather restrictive and lacks the plausible justification. Indeed, among all arithmetic operations there is exactly one, which is particularly easy for the human beings. This is the computation of the maximum among two values. Therefore, we have good chances that this operation is included somehow in the structure of utility functions. In this case, the utility function becomes non-differentiable and the set of its marginal utilities has nothing common with its actual (sub)gradient.

Finally, let us discuss the last element of Arrow-Debreu model, the budget constraint. Due to the pioneering contribution of Walras (at least), the notion of budget constraint is so old, natural and common, that there was seemingly no attempt to criticize its actual role in the hidden difficulties related to this model. It is well known that one of the main drawbacks of Arrow-Debreu model is the multiplicity of equilibrium prices. Moreover, these prices are not endowed with any “functional” characterization, which could help us in approaching them in the real-life economic models. In fact, equilibrium prices are characterized as zeros of the excess demand correspondence. Under well-known assumptions on the consumers’ utilities, the excess demand correspondence becomes continuous or even differentiable function (e.g., [8]). As shown in [6], its zeros are locally unique and their number is generically finite. Moreover, the uniqueness of equilibrium prices is known to hold only under very particular assumptions, e.g. in case of gross substitutes [3]. The absence of uniqueness is a serious drawback for any applied theory. It destroys a hope for a reliable prediction of the future. Moreover, very often it forces to switch from the static to dynamic models, which by the order of magnitude are more complicated. And this is not only a theoretical difficulty. If our model is relevant, nonuniqueness implies that the agents in the real life also can be attracted by different equilibrium states. Hence, we cannot say too much about the final state of our system.

Coming back to the solvability issues, recall that in Optimization Theory there are good reasons to believe that the maximal class of tractable models is formed by problems with *convex structure* (e.g. see discussion in Section 2.1 in [14]). Some of the most general problems of this type are the *saddle-point problems*, where we have a set of primal variables (for minimization) and dual variables (for maximization). The potential function of this problem must be convex in primal variables and concave in the dual ones. In principle, this framework could be used for describing the economic equilibrium with prices as primal variables and production/consumption volumes as the dual ones. Unfortunately, for its solvability it is necessary to have *separable* convex constraints for primal and dual variables. This is exactly the place where the budget constraint destroys any hope for existence of an efficient strategy, which could approach the equilibrium prices and consumption/production volumes.

The above mentioned drawbacks of Arrow-Debreu model were served as the main motivations for our research. The results will be presented in several subsequent papers, which describe the way of equilibrium functioning of *capitalistic economy*. The main elements of our theory are as follows.

- We introduce and explain a natural consumption strategy for buying the daily goods, which can be seen as a numerical method for minimizing some special disutility function. This function is *derived* from the natural consumption cycle:

$$\begin{array}{ccc} \boxed{\begin{array}{c}\textit{Deciding on the budget}\\ \textit{for single shopping.}\end{array}} & \Rightarrow & \boxed{\begin{array}{c}\textit{Buying products with the}\\ \textit{best quality/price ratio.}\end{array}} \\ \Uparrow & & \Downarrow \\ \boxed{\begin{array}{c}\textit{Observation of the}\\ \textit{results.}\end{array}} & \Leftarrow & \boxed{\begin{array}{c}\textit{Consumption of the}\\ \textit{products bought.}\end{array}} \end{array} \tag{1}$$

  We argue that the implementation of this cycle by a customer needs only regular updates of some subconscious estimates (individual *prices for qualities*), which allow to fight for a *reasonable price* of the products available on the market. The corresponding average consumption pattern is inserted in our general model.

- We assume that the producer has enough computational power and informational support in order to form the optimal production plan using the current market prices for row materials and production items. His actions are restricted only by the technological constraints.

- In our model, the production and consumption volumes, the income values, and expenses are considered as *constant flows*. This means that we get the same amount of corresponding objects in each standard interval of time (say, one week). Therefore, if the income of a person or a firm during this interval is greater than the expenses, then he/she can ensure a constant *rate of growth* of the own capital. If the income is strictly less than expenses, then the firm/person must leave the market. This is true both for producers (bankruptcy), and for consumers (emigration from this market).

- If the regular income is equal to the regular expenses, then this is a marginal case, which we call *poverty*. This unpleasant state is feasible both for firms and for individuals. In this case, their production/consumption volume is not maximal. They implement only a fraction of their potential activity. A hidden consequence of this marginal behavior is a possibility to balance the market. From the technical point of view, introduction of poverty allows lifting the budget constraint into a joint potential function, ensuring the convexity of the whole model.
- Our model admits a joint potential function, which represents the rate of growth of the capital of whole society. We call it the *Total Excessive Revenue* (TER). This function has the prices and salaries as primal variables (for minimization), and the production and consumption volumes as dual variables (for maximization). The main theorem of capitalistic economy reads then:

  Equilibrium set of prices and salaries minimizes the total excessive revenue of the whole society.
- The necessary and sufficient condition for existence an equilibrium in capitalistic economy is an existence of certain sub-economy, which can survive autonomously.
- Our model can be seen as a convex-concave game. This implies the existence of numerous simple algorithmic schemes for approaching its solution. We will show that some of these adjustment processes have natural behavioral interpretations. This can be seen as a confirmation of an intuitive evolutionary principle:

  Winning economical systems provide more opportunities for rational behavior.

At the same time, the existence of convex-concave potential for the market implies that the capitalistic economy is intrinsically stable. Clear goals of the participants and predictable behavior of the prices result in a fast elimination of consequences caused by unexpected shocks and perturbations.

In this paper we introduce the main elements of our model, prove existence and welfare theorems. In the next paper we are going to consider different extensions of our model on the macroscopic level, at which we can observe the role of money, tax regulations, budget of the state, etc. We will discuss which elements can be easily incorporated in the model preserving its convexity, and which of them do not possess this property. The latter elements should be considered as dangerous since they introduce significant instability in the system. Last paper of the series will be devoted to algorithmic aspects.

Notation. Our notation is quite standard. We denote by $\mathbb{R}^n$ the space of n-dimensional column vectors $x = (x^{(1)}, \ldots, x^{(n)})^T$, and by $\mathbb{R}^n_+$ the set of all vectors with nonnegative components. For x and y from $\mathbb{R}^n$, we introduce the standard scalar product

$$\langle x, y \rangle \quad = \quad \sum_{i=1}^{n} x^{(i)} y^{(i)}.$$

We denote by $e_j \in \mathbb{R}^k$ the jth coordinate vector of the corresponding vector space. The same notation is used for different spaces, which are always determined by the context. Finally, $\mathbb{R}^{k \times m}$ denotes the space of $k \times m$-matrices, and $(a)_+$ denotes the positive part of value $a \in \mathbb{R}$: $(a)_+ = \max\{a, 0\}$.

2 Excessive revenue model

Let us introduce the *excessive revenue model* of capitalistic market with multiple goods. For that, we need to describe the behavior of producers and consumers. We assume that all participants of the market are rational. This means that, given a set of fixed prices, they maximize their returns and minimize their expenditures by adjusting production and consumption patterns, which are compatible with their needs and technological constraints. Then, we define the equilibrium production and consumption flows, which can be balanced by some equilibrium prices. We prove that these prices minimize the total excessive revenue of the whole society.

Our model can be applied to different types of markets with different relations between participants. In order to introduce the main ideas, we start from the model of the simplest local market describing, say, some tourist services. In this situation, we can think about economic relations between the local food producers, restaurants and hotels with tourists, which have a constant income, independent on this market. This level of details is sufficient for demonstrating the main principles of construction of our model. Later, in Section 5 we consider a simple extension, which gives a possibility for consumers be employed by producers.

2.1 Producers

Consider a market with K producers, which are able to produce n different goods. The k-th producer has to choose the vector of production volumes $u_k \in \mathcal{U}_k \subset \mathbb{R}^n_+$, where $\mathcal{U}_k$ is the *maximal* technological set. We assume that the set $\mathcal{U}_k$ is closed and convex. In the majority of realistic situations, this set is bounded.

Given a vector of prices $p \in \mathbb{R}^n_+$, the producer's yield is then $\langle p, u_k \rangle$. At the same time, we distinguish the following production costs of the producer.

(a) *Internal cost* comes from buying the necessary compounds, available on the market. The internal technological matrix $A_k \in \mathbb{R}^{n\times n}_+$ defines the amounts of compounds employed by the k-th producer. Its column $A_k e_j \in \mathbb{R}^n$ is formed by the required volumes of corresponding goods, which are necessary for producing one unit of product j, $j = 1, \ldots, n$, by producer k. The internal cost is then given by $\langle p, A_k u_k \rangle$.

(b) *External cost* comes from the necessity of buying some row materials outside the local market. There are r different resources with corresponding vector of prices $y \in \mathbb{R}^r_+$. The external technological matrix $R_k \in \mathbb{R}^{r\times n}$ describes the resources required by the production process of k-th producer. Its column $R_k e_j \in \mathbb{R}^r$ represents the volumes of corresponding resources needed for producing one unit of good j by k-th producer. The external cost is then given by $\langle y, R_k u_k \rangle$. We assume that the total amounts of available external resources are bounded by some upper limits stored in vector $b \in \mathbb{R}^r_+$. Thus, the monetary value of available external resources is $\langle y, b \rangle$.

(c) *Production cost* includes expenditures, which are necessary for producing unit volumes of the products. We put them into the vector $c_k \in \mathbb{R}^n_+$. Thus, the production cost is given by $\langle c_k, u_k \rangle$.

(d) *Fixed cost* of maintaining the technological set $\mathcal{U}_k$, denoted by $\varkappa_k \equiv \varkappa_k(\mathcal{U}_k) \in \mathbb{R}_+$. It can include the interest paid to the bank, different charges for renting the equipment, land use, etc. We assume its homogeneity:

$$\varkappa_k(\alpha \mathcal{U}_k) = \alpha \varkappa_k(\mathcal{U}_k), \quad \alpha \in [0,1]. \tag{2}$$

We consider the production processes with single technology. Generalization of the model onto the case of multiple technologies is straightforward, and we omit it for the sake of simplicity.

Denote by $\pi = (p, y) \in \mathbb{R}^{n+r}$ the system of existing prices. We assume that our producers are rational and that they have full informational support. Therefore, in order to find the optimal production volume u_k, the k-th producer just needs to solve the following maximization problem:

$$PB_k(\pi) \stackrel{\mathrm{def}}{=} \max_{u_k \in \mathcal{U}_k} \left[\, \langle p - c_k, u_k \rangle - \langle y, R_k u_k \rangle - \langle p, A_k u_k \rangle \, \right]. \tag{3}$$

Clearly, this function is convex in π as the maximum of linear functions. Denote by $\mathcal{U}_k^* \equiv \mathcal{U}_k^*(\pi)$ the set optimal solutions of the maximization problem in (3). Thus, the k-th producer's revenue is

$$PR_k(\pi) \stackrel{\mathrm{def}}{=} PB_k(\pi) - \varkappa_k = \langle p - c_k - R_k^T y - A_k^T p, u_k^* \rangle - \varkappa_k, \quad u_k^* \in \mathcal{U}_k^*(\pi).$$

It can happen, that the k-th producer can not cover his fixed cost $\varkappa_k$. Recall that we are speaking about repetitive production processes, which generate the constant flows of goods. Producer with regular negative revenue *cannot survive* at such market. In order to treat this situation, it is convenient to introduce the *excessive revenue* of k-th producer:

$$EPR_k(\pi) \stackrel{\mathrm{def}}{=} (\, PR_k(\pi) \,)_+ = \max_{u_k \in \mathcal{U}_k} \left(\langle p - c_k - R_k^T y - A_k^T p, u_k \rangle - \varkappa_k \right)_+, \tag{4}$$

which reflect the fact that a producer with negative revenue simply disappears from the model. This is a convex function in π.

Note that in many situations the maximal technological set $\mathcal{U}_k$ can be modelled as follows. We assume that k-th producer has m_k production facilities. The capacities of these facilities are given by vector $f_k \in \mathbb{R}^{m_k}_+$. Then the allocation matrix $G_k \in \mathbb{R}^{m_k \times n}$ describes the volumes of the facilities required in the production process. Its column $G_k e_j \in \mathbb{R}^{m_k}$ represents the volumes of the facilities needed for producing one unit of good j by k-th producer. Then,

$$\mathcal{U}_k = \left\{ u_k \in \mathbb{R}^n_+ : \; G_k u_k \le f_k \right\}. \tag{5}$$

In principle, such a representation allows to define the *shadow prices* for equipment as the Lagrange multipliers for corresponding linear inequality constraints. However, in this paper we do not discuss such issues.

2.2 Consumers

Consider a market with L consumers. The i-th consumer has to decide on his regular consumption volumes of different goods, which we store in the vector $v_i \in \mathbb{R}^n_+$. Given a vector of prices $p \in \mathbb{R}^n_+$, the consumer's expenditure is then $\langle p, v_i \rangle$.

Let us describe now how the consumer selects the products. We assume that the consumers judge the goods in accordance to the presence in them of d different *qualities*. The personal estimates of the volumes of these qualities are stored in the matrix $Q_i \in \mathbb{R}^{d\times n}$. Here, the value $(Q_i e_j)^{(k)}$ is the personal estimate of i-th consumer for the amount of quality k in one unit of the good j.

Further, we assume that each customer has some standards for regular consumptions of certain amounts of qualities during the standard interval of time. We store these standards in vectors $\sigma_i \in \mathbb{R}^d_+$. Then, the feasible set of consumption flows for consumer i is $\mathcal{V}_i := \left\{ v_i \in \mathbb{R}^n_+ : \, Q_i v_i \geq \sigma_i \right\}$.

Now we introduce our main assumption. In order to find the optimal consumption pattern v_i, the i-th consumer needs to minimize his regular expenditure:

$$CE_i(\pi) := \min_{v_i \in \mathcal{V}_i} \langle p, v_i \rangle. \tag{6}$$

This function is concave in π as minimum of linear functions. Denote by $\mathcal{V}_i^* \equiv \mathcal{V}_i^*(\pi)$ the set of optimal solutions to problem (6).

Our main assumption is that the consumers *are able to solve this problem*. Of course, this is a very strong assumption. As it was already mentioned in Introduction, the dimension of this problem can be very high. It is indeed difficult to believe that a normal human being is able to find somehow a nearly optimal solution to (6). However, this very important question has indeed a positive answer. In Section 6 we argue that this consumption pattern can be approached by a natural consumption behavior, implemented in a form of the regular shopping cycle (1).

Denote by $\tau_i \in \mathbb{R}_+$ the budget of i-th consumer. Then the revenue of i-th consumer can be computed as

$$CR_i(\pi) := \tau_i - CE_i(\pi) = \tau_i - \langle p, v_i^* \rangle, \quad v_i^* \in \mathcal{V}_i^*.$$

If the regular budget of i-th consumer is not sufficient for ensuring the standards σ_i, then the consumer must leave the market (emigration). Thus, similarly to the producers, we can introduce the excessive revenue of consumers:

$$ECR_i(\pi) := (\, CR_i(\pi) \,)_+ \, .$$

Clearly, this function is convex in π, and it can be represented as follows:

$$ECR_i(\pi) = \max_{v_i \in \mathcal{V}_i} \left(\tau_i - \langle p, v_i \rangle \right)_+ . \tag{7}$$

2.3 Equilibrium market flows

In our model we need to introduce the levels of participation of the agents in the market activity. These levels are described by certain coefficients between zero and one. They depend on the regular revenues of the agents as follows.

If the revenue of an agent is strictly negative, then the participation level is zero. In this case, a producer can not cover his fixed costs by the return, and a consumer can not afford his purchases by the available budget. In both cases, the agent must leave the market.

If the revenue of an agent is strictly positive, then his participation level is one. Thus, the agent generates an excessive revenue due to the return or budget surplus. This case corresponds to the full (maximal) involvement of the agent in the market activity.

If the revenue of an agent is zero, then the situation is more delicate. For producer, this means that his activity, even within the maximal technological set $\mathcal{U}_k$ does not generate any profit. The situation is not changing if the production activities (the set $\mathcal{U}_k$) will be proportionally reduced by a factor $\alpha_k \in [0,1]$ (recall the assumption (2)). Thus, it is natural to admit that in this marginal situation the producer can work with a reduced technological set $\alpha_k \mathcal{U}_k$. The particular value of α_k depends on the individual history of successes and failures for this producer.

If the revenue of a consumer is zero, then again, there is no special reason to allocate all the budget τ_i to this expensive market. The consumer can decide to spend here only a part of it, namely $\beta_i \tau_i$ with some $\beta_i \in [0,1]$, which is sufficient to cover the pattern $\beta_i \sigma_i$ for the consumption of qualities (this does not change the zero level of the excessive revenue). The remaining part $(1-\beta_i)\tau_i$ of the budget can be used then at another markets. Again, the appropriate value of this coefficient can be found by some adjustment processes. We will see later, that the presence of these coefficients is often necessary for balancing the production and consumption volumes on the market.

Fractional value of the participation level can be seen as an indication (and measure) of *poverty*. Note that this can happen both with producers and consumers.

The participation levels of the agents can be included directly in definitions of the excessive revenues:

$$\begin{array}{rcccl} EPR_k(\pi) & = & (PR_k(\pi))_+ & = & \max\limits_{\alpha_k \in [0,1]} \left[\alpha_k PR_k(\pi) \right], \\ ECR_i(\pi) & = & (CR_i(\pi))_+ & = & \max\limits_{\beta_i \in [0,1]} \left[\beta_i CR_i(\pi) \right]. \end{array}$$

In particular, if $\alpha_k \in [0,1]$ is the participation level of k-th producer, then

$$\alpha_k PR_k(\pi) = \max_{\widetilde{u}_k \in \alpha_k \mathcal{U}_k} \left[\langle p - c_k, \widetilde{u}_k \rangle - \langle y, R_k \widetilde{u}_k \rangle - \langle p, A_k \widetilde{u}_k \rangle - \alpha_k \varkappa_k(\mathcal{U}_k) \right].$$

Thus, the actual participation $\widetilde{u}_k \equiv \widetilde{u}_k(\alpha_k)$ of the producer in the market activity is only an α_k-fraction of its maximal production abilities: $\widetilde{u}_k = \alpha_k u_k^*$ with some $u_k^* \in \mathcal{U}_k^*$.

Similarly, let $\beta_i \in [0,1]$ be participation level of i-th consumer. Then, he satisfies just a β_i-fraction of his desired life standards σ_i by spending the corresponding part of the budget τ_i:

$$\beta_i CR_i(\pi) = \max_{\widetilde{v}_i \in \beta_i \mathcal{V}_i} \left[\beta_i \tau_i - \langle p, \widetilde{v}_i \rangle \right].$$

Thus, the actual consumption pattern of ith consumer at this market is $\widetilde{v}_i = \widetilde{v}_i(\beta_i) = \beta_i v_i^*$ with some $v_i^* \in \mathcal{V}_i^*$.

Now we can give a formal definition of proper participation levels.

Definition 1 *For a given system of prices* $\pi = (p, y) \in \mathbb{R}^{n+r}_+$ *and a system of* tentative market flows

$$F_0 = \left(\left\{ u_k^0 \right\}_{k=1}^K, \left\{ v_i^0 \right\}_{i=1}^L \right) \in \Omega \stackrel{\mathrm{def}}{=} \prod_{k=1}^K \mathcal{U}_k \times \prod_{i=1}^L \mathcal{V}_i,$$

the system of participation levels $\gamma = \left(\{\alpha_k\}_{k=1}^K, \{\beta_i\}_{i=1}^L \right) \in [0,1]^{K+L}$ *is called* proper *(with respect to* π *and* F_0*) if it satisfies the following conditions:*

$$\alpha_k = \begin{cases} 1, & \text{if } \langle p - c_k - R_k^T y - A_k^T p, u_k^0 \rangle > \varkappa_k(\mathcal{U}_k), \\ 0, & \text{if } \langle p - c_k - R_k^T y - A_k^T p, u_k^0 \rangle < \varkappa_k(\mathcal{U}_k), \end{cases}$$

$$\beta_i = \begin{cases} 1, & \text{if } \langle p, v_i^0 \rangle < \tau_i, \\ 0, & \text{if } \langle p, v_i^0 \rangle > \tau_i. \end{cases}$$

Such a triple (π, F_0, γ) *defines a* real market flow $F_r = \left(\left\{ \alpha_k u_k^0 \right\}_{k=1}^K, \left\{ \beta_i v_i^0 \right\}_{i=1}^L \right)$. □

Now we can define the equilibrium market flows. Let $F_r = \left(\{\widetilde{u}_k\}_{k=1}^K, \{\widetilde{v}_i\}_{i=1}^L \right)$ be a real market flow defined by some triple (π, F_0, γ). We define two balancing conditions.

(B1) **The market of goods is balanced.** Namely, the consumption volumes never exceed the volumes of production, and the market of goods with positive prices ($p^{(j)} > 0$) are perfectly balanced:

$$\sum_{k=1}^K \widetilde{u}_k^{(j)} = \sum_{k=1}^K (A_k \widetilde{u}_k)^{(j)} + \sum_{i=1}^L \widetilde{v}_i^{(j)}.$$

The left-hand side of this equation represents the production volume of j-th good. Its right-hand side accounts the producers' supplies employed for the production, and the consumption volumes.

(B2) **The market of resources is balanced.** This means that the external resources used for production never exceed their limiting bounds. Moreover, the market of resources with positive prices ($y^{(j)} > 0$) is balanced:

$$\sum_{k=1}^K (R_k \widetilde{u}_k)^{(j)} = b^{(j)}.$$

The left-hand side of this equation aggregates the needs of producers in j-th resource. The right-hand side represents the upper limits for corresponding resources.

Definition 2 *We say that the set* $\pi \in \mathbb{R}_+^{n+r}$ *is the set of* equilibrium prices *if there exists a tentative flow*

$$F^* \in \Omega_*(\pi) \stackrel{\text{def}}{=} \prod_{k=1}^K \mathcal{U}_k^*(\pi) \times \prod_{i=1}^L \mathcal{V}_i^*(\pi)$$

and a proper system of participation levels γ *such that the corresponding real market flow* F_r^* *satisfies the balancing conditions (B1) and (B2).*

In this case F_r^* *is called the* equilibrium market flow. □

3 Existence and characterization theorems

At equilibrium, the characteristic behavior of producers and consumers is very simple: all of them are trying to maximize their revenues (see (4) and (7)). At the same time, the origin of *equilibrium prices* is not so evident[1]. For its characterization, let us introduce the notion of *Total Excessive Revenue* of the market, which is just the sum of excessive revenues of all agents:

$$\begin{aligned} TER(\pi) \;&\stackrel{\mathrm{def}}{=}\; \sum_{k=1}^{K} EPR_k(\pi) + \sum_{i=1}^{L} ECR_i(\pi) + \langle b, y\rangle \\ &= \sum_{k=1}^{K} \max_{u_k\in\mathcal{U}_k} \left(\langle p - c_k - R_k^T y - A_k^T p, u_k\rangle - \varkappa_k\right)_+ \\ &\quad + \sum_{i=1}^{L} \max_{v_i\in\mathcal{V}_i} \left(\tau_i - \langle p, v_i\rangle\right)_+ + \langle b, y\rangle. \end{aligned} \tag{8}$$

In this expression, $\langle b, y\rangle$ is the cost of all available external resources. It represents the revenue of the external providers. Note that function $TER(\cdot)$ is convex since it is a sum of convex functions.

Theorem 1 *The set $\pi_* = (p_*, y_*) \in \mathbb{R}^{n+r}_+$ is the set of equilibrium prices if and only if it solves the following convex minimization problem:*

$$\min_{\pi\in R^{n+r}_+} TER(\pi). \tag{9}$$

Proof:
1. Assume that the set $\pi_* = (p_*, y_*)$ is formed by equilibrium prices. Then, in view of Definition 2, there exist a tentative flow $F_* = \left(\{u_k^*\}_{k=1}^K, \{v_i^*\}_{i=1}^L\right) \in \Omega_*(\pi_*)$, and a proper participation level $\gamma^* = \left(\{\alpha_k^*\}_{k=1}^K, \{\beta_i^*\}_{i=1}^L\right)$, such that the real market flow

$$F_r^* = \left(\{\widetilde{u}_k = \alpha_k^* u_k^*\}_{k=1}^K, \{\widetilde{v}_i = \beta_i^* v_i^*\}_{i=1}^L\right),$$

satisfies the conditions (B1) and (B2).

Denote $\xi_p^* = \sum_{k=1}^K (\widetilde{u}_k - A_k\widetilde{u}_k) - \sum_{i=1}^L \widetilde{v}_i$, and $\xi_y^* = b - \sum_{k=1}^K R_k\widetilde{u}_k$. Due to the balances for goods and resources (B1)-(B2), vector $\xi^* \stackrel{\mathrm{def}}{=} \left(\xi_p^*, \xi_y^*\right)$ has non-negative components. Moreover, the components of this vector, which correspond to the positive prices, are equal to zero. Hence, for all $\pi = (p, y) \geq 0$ we have $\langle \xi^*, \pi - \pi_*\rangle \geq 0$. Further, in view of (4) and (7), $\xi^* \in \partial TER(\pi_*)$. Since TER is convex in π, for all $\pi = (p, y) \geq 0$ we have:

$$TER(\pi) - TER(\pi_*) \geq \langle \xi^*, \pi - \pi_*\rangle \geq 0.$$

Thus, π_* minimizes the Total Excessive Revenue.

[1] Remember the famous "invisible hand" of the market by Adam Smith.

2. Assume that the set of prices $\pi_* = (p_*, y_*) \geq 0$ is optimal for problem (9). Then there exists $\xi^* \in \partial TER(\pi_*)$ such that

$$\langle \xi^*, \pi - \pi_* \rangle \geq 0, \quad \forall \pi \geq 0.$$

Considering $\pi = 0$ and $\pi = 2\pi_*$, we conclude that $\langle \xi^*, \pi_* \rangle = 0$. Consequently, $\xi^* \geq 0$. Moreover,

$$\pi_*^{(j)} = 0 \quad \Rightarrow \quad \xi^{*(j)} \geq 0,$$

$$\pi_*^{(j)} > 0 \quad \Rightarrow \quad \xi^{*(j)} = 0.$$

It remains to note that in view of representations (4) and (7) we have $\xi^* = (\xi_p^*, \xi_y^*)$ with

$$\begin{aligned} \xi_p^* &= \sum_{k=1}^{K} \alpha_k^* \left(u_k^* - A_k u_k^*\right) - \sum_{i=1}^{L} \beta_i^* v_i, \\ \xi_y^* &= b - \sum_{k=1}^{K} \alpha_k^* R_k u_k^*, \end{aligned}$$

where $u_k^* \in \mathcal{U}_k^*$, $k = 1, \dots, K$, $v_k^* \in \mathcal{V}_i^*$, $i = 1, \dots, L$, and the system of participation levels $\gamma^* = \left(\{\alpha_k^*\}_{k=1}^K, \{\beta_i^*\}_{i=1}^L\right) \in [0,1]^{K+L}$ is proper for π_* and the tentative flow $F_* = \left(\{u_k^*\}_{k=1}^K, \{v_i^*\}_{i=1}^L\right)$. Therefore, the conditions (B1) and (B2) are satisfied by the real market flow $F_r^* = \left(\{\alpha_k^* u_k^*\}_{k=1}^K, \{\beta_i^* v_i^*\}_{i=1}^L\right)$. Hence, the set π_* is formed by equilibrium prices. □

Let us present some conditions for existence of an equilibrium in the market.

Definition 3 *A market is called* productive *if there exist subsets of producers $\mathcal{K} \subset \{1, \dots, K\}$ and consumers $\mathcal{L} \subset \{1, \dots, L\}$, such that the corresponding production and consumption flows*

$$\left(\{\bar{u}_k\}_{k \in \mathcal{K}}, \{\bar{v}_i\}_{i \in \mathcal{L}}\right) \in \prod_{k \in \mathcal{K}} \mathcal{U}_k \times \prod_{i \in \mathcal{L}} \mathcal{V}_i$$

establish positive balances for production of goods and consumption of external resources:

$$\begin{aligned} \sum_{k \in \mathcal{K}} \bar{u}_k &> \sum_{k \in \mathcal{K}} A_k \bar{u}_k + \sum_{i \in \mathcal{L}} \bar{v}_i, \\ b &> \sum_{k \in \mathcal{K}} R_k \bar{u}_k. \end{aligned} \tag{10}$$

Theorem 2 *At the productive markets, the level sets of function $TER(\cdot)$ are bounded. This implies existence of equilibrium prices.*

Proof:

Denote $\bar{\xi}_p = \sum\limits_{k\in\mathcal{K}} \bar{u}_k - \sum\limits_{k\in\mathcal{K}} A_k\bar{u}_k - \sum\limits_{i\in\mathcal{L}} \bar{v}_i$, and $\bar{\xi}_y = b - \sum_{k\in\mathcal{K}} R_k\bar{u}_k$. For all $\pi = (p,y) \geq 0$ we have

$$\begin{array}{rcl} TER(\pi) & = & \sum\limits_{k=1}^{K} [PR_k(\pi)]_+ + \sum\limits_{i=1}^{L} [CR_i(\pi)]_+ + \langle y, b\rangle \\ & \geq & \sum\limits_{k\in\mathcal{K}} [PR_k(\pi)]_+ + \sum\limits_{i\in\mathcal{L}} [CR_i(\pi)]_+ + \langle y, b\rangle \\ & \geq & \sum_{k\in\mathcal{K}} PR_k(\pi) + \sum_{i\in\mathcal{L}} CR_i(\pi) + \langle y, b\rangle \\ & \geq & \sum\limits_{k\in\mathcal{K}} (\langle p, \bar{u}_k\rangle - \langle y, R_k\bar{u}_k\rangle - \langle p, A_k\bar{u}_k\rangle - \varkappa_k) + \sum\limits_{i\in\mathcal{L}} (\tau_i - \langle p, \bar{v}_i\rangle) + \langle y, b\rangle \\ & = & -\sum\limits_{k\in\mathcal{K}} (\varkappa_k + \langle c_k, \bar{u}_k\rangle) + \sum\limits_{i\in\mathcal{L}} \tau_i + \left\langle \left(\bar{\xi}_p, \bar{\xi}_y\right), \pi\right\rangle . \end{array}$$

Since $\left(\bar{\xi}_p, \bar{\xi}_y\right) > 0$, the intersection of the level sets of function TER with $\mathbb{R}^{n+r}_+$ is bounded. Hence, problem (9) is solvable. Thus, the existence of equilibrium prices follows from Theorem 1. $\square$

Remark 1 *Note that condition* (10) *coincides with the standard productivity condition in Leontief's input-output economic model of a closed economy system (see [10]) only if* $A_k = A$ *for all* $k \in \mathcal{K}$. *In any case, our condition is applicable to a more general situation.*

Remark 2 *If all technological sets* $\mathcal{U}_k$ *are defined by (5), then the function* $TER(\cdot)$ *is piece-wise linear. In this case, since it is nonnegative, the optimal set of problem (9) is nonempty. However, for its boundedness we still need to assume productivity of the market.*

We need to introduce also some additional assumptions in order to guarantee that our market indeed works. Namely, we need to ensure that the optimal solution π_* of the problem (9) in not at the origin. For that, we introduce the following condition rejecting the *Zero-Cost Production* (ZCP):

$$\text{If } \alpha_k\varkappa_k + \langle c_k, u_k\rangle = 0 \text{ with } u_k \in \alpha_k\mathcal{U}_k \text{ and } \alpha_k \in [0,1], \text{ then } u_k = 0. \tag{11}$$

This condition is automatically satisfied for $\varkappa_k > 0$. If $\varkappa_k = 0$, then (11) implies that for the kth producer there is no nonzero production plan with zero production cost. Recall that

$$EPR_k(\pi) = \max_{\alpha_k, u_k} [\langle p - c_k - R_k^T y - A_k^T p, u_k\rangle - \alpha_k\varkappa_k : \ \alpha_k \in [0,1],\ u_k \in \alpha_k\mathcal{U}_k].$$

Therefore, condition (11) implies that $\partial EPR_k(0) = \{0\}$.

Assume now that the income τ_i of ith consumer is positive. Since

$$ECR_i(\pi) = \max_{\beta_i, v_i} [\beta_i\tau_i - \langle p, v_i\rangle : \ \beta_i \in [0,1],\ v_i \in \beta_i\mathcal{V}_i],$$

we conclude that $\partial ECR_i(0) = (-\mathcal{V}_i, 0)$. Thus, we have proved the following statement.

Lemma 1 *Let all producers satisfy ZCP-condition, and the incomes of all consumers are positive. Then*

$$\partial TER(0) \quad = \quad \left(-\sum_{i=1}^{L} \mathcal{V}_i, b \right). \tag{12}$$

Corollary 1 *Existence of a consumer with nonzero life standard is sufficient for having* $\pi_* \neq 0$.

Proof:
Indeed, assume that $\pi_* = 0$. In view of the first-order optimality conditions, there exists $\xi^* \in \partial TER(0)$ such that

$$\langle \xi^*, \pi \rangle \quad \geq \quad 0 \quad \forall \pi \geq 0.$$

Hence, $\xi^* = (-\sum\limits_{i=1}^{L} v_i^*, b) \geq 0$ for some $v_i^* \in \mathcal{V}_i$. Therefore, all $v_i^* = 0$, implying zero life standards for all consumers. □

It is interesting that the last statement is formulated only in terms of consumption standards. This confirms the primary role of demand in generating supply.

4 Properties of equilibrium market flows

Let us present a welfare theorem for equilibrium market flow (compare with welfare theorem for Walrasian equilibrium, see [2]). We are going to prove that any equilibrium market flow is Pareto optimal. This means that no consumers or producers can improve his gain (excessive revenue) without worsening the gain of some others. Let us start from the definition of feasible market flows.

Definition 4 *The real market flow is called* feasible *if it satisfies the balancing conditions (B1) and (B2).* □

Definition 5 *A feasible market flow F_*, defined by the triple*

$$\left(\pi_* = (p_*, y_*), F_0^* = \left(\left\{ u_k^0 \right\}_{k=1}^{K}, \left\{ v_i^0 \right\}_{i=1}^{L} \right), \gamma_* \right),$$

is called Pareto optimal *if there is no feasible market flow $\hat{F}$ defined by another triple*

$$\left(\hat{\pi} = (\hat{p}, \hat{y}), \hat{F}_0 = \left(\{ \hat{u}_k \}_{k=1}^{K}, \{ \hat{v}_i \}_{i=1}^{L} \right), \hat{\gamma} \right)$$

such that all inequalities

$$\begin{array}{rcl} (\langle \hat{p} - c_k - R_k^T \hat{y} - A_k^T \hat{p}, \hat{u}_k \rangle - \varkappa_k)_+ & \geq & (\langle p_* - c_k - R_k^T y_* - A_k^T p_*, u_k^0 \rangle - \varkappa_k)_+, \\ & & k = 1 \ldots K, \\ (\tau_i - \langle \hat{p}, \hat{v}_i \rangle)_+ & \geq & (\tau_i - \langle p_*, v_i^0 \rangle)_+, \; i = 1 \ldots L, \\ \langle b, \hat{y} \rangle & \geq & \langle b, y_* \rangle, \end{array} \tag{13}$$

are satisfied, and one of them is strict.

Note that we define Pareto optimality with respect to excessive revenues. In our model they play a role of objective functions of the agents.

Theorem 3 *Any equilibrium market flow is Pareto optimal.*

Proof:
Using notation of Definition 5 assume that the set π_* is formed by equilibrium prices, and F_* is the corresponding equilibrium market flow. Assume that the inequalities (13) are all valid for some feasible market flow $\hat{F}$ defined by the triple $(\hat{\pi}, \hat{F}_0, \hat{\gamma})$. And let at least one of these inequalities be strict. For $\pi = (p, y) \in \mathbb{R}^{n+r}$ and $F \in \Omega$, define the function

$$\varphi(\pi, F) = \sum_{k=1}^{K} (\langle p - c_k - R_k^T y - A_k^T p, u_k \rangle - \varkappa_k)_+ + \sum_{i=1}^{L} (\tau_i - \langle p, v_i \rangle)_+ + \langle b, y \rangle.$$

In view of our assumption, $\varphi(\hat{\pi}, \hat{F}_0) > \varphi(\pi_*, F_*)$. Since π_* is the set of equilibrium prices, in view of Theorem 1 and definitions (4), (7) we have:

$$\varphi(\pi_*, F_*) = \min_{\pi \geq 0} \max_{F \in \Omega} \varphi(\pi, F) = \max_{F \in \Omega} \min_{\pi \geq 0} \varphi(\pi, F) \geq \min_{\pi \geq 0} \varphi(\pi, \hat{F}_0).$$

It remains to note that the balancing conditions (B1) and (B2) for the flow $\hat{F}$ are exactly the necessary and sufficient characterization of point $\hat{\pi}$ as the optimal solution to the latter minimization problem. Therefore, $\varphi(\pi_*, F_*) \geq \varphi(\hat{\pi}, \hat{F}_0)$. This is a contradiction. □

In view of Theorem 1, equilibrium prices minimize the total excessive revenue. Let us prove a very intuitive result that its optimal value is equal to the difference of the sum of the real budgets of consumers and the sum of the real expenses of producers.

Theorem 4 *Let $\pi_* = (p_*, y_*)$ be the equilibrium system of prices, supported by the tentative flow F_0^* and a proper system of participation levels $\gamma_* = \left(\{\alpha_k^*\}_{k=1}^K, \{\beta_i^*\}_{i=1}^L \right)$. Then $TER(\pi_*) = \sum\limits_{i=1}^{L} \beta_i^* \tau_i - \sum\limits_{k=1}^{K} \alpha_k^* (\varkappa_k + \langle c_k, u_k^* \rangle) \geq \langle b, y_* \rangle$.*

Proof:

Denote by $F_0^* = \left(\{u_k^*\}_{k=1}^K, \{v_i^*\}_{i=1}^L \right)$ the tentative flow pattern. Then

$$\begin{aligned}
TER(\pi_*) &= \sum_{k=1}^{K} \left(\langle p_* - c_k, u_k^* \rangle - \langle y^*, R_k u_k^* \rangle - \langle p_*, A_k u_k^* \rangle - \varkappa_k \right)_+ \\
&\quad + \sum_{i=1}^{L} \left(\tau_i - \langle p_*, v_i^* \rangle \right)_+ + \langle b, y_* \rangle \\
&= \sum_{k=1}^{K} \alpha_k^* \left(\langle p_* - c_k, u_k^* \rangle - \langle y_*, R_k u_k^* \rangle - \langle p_*, A_k u_k^* \rangle - \varkappa_k \right) \\
&\quad + \sum_{i=1}^{L} \beta_i^* \left(\tau_i - \langle p_*, v_i^* \rangle \right) + \langle b, y_* \rangle \\
&= \left\langle p_*, \sum_{k=1}^{K} \alpha_k^* (u_k^* - A_k u_k^*) - \sum_{i=1}^{L} \beta_i^* v_i^* \right\rangle + \left\langle y_*, b - \sum_{k=1}^{K} \alpha_k^* R_k u_k^* \right\rangle \\
&\quad - \sum_{k=1}^{K} \alpha_k^* (\varkappa_k + \langle c_k, u_k^* \rangle) + \sum_{i=1}^{L} \beta_i^* \tau_i^* + \langle b, y_* \rangle.
\end{aligned}$$

In view of balancing equations (B1) and (B2), we have

$$\left\langle p_*, \sum_{k=1}^{K} \alpha_k^* (u_k^* - A_k u_k^*) - \sum_{i=1}^{L} \beta_i^* v_i^* \right\rangle = 0, \quad \left\langle y_*, b - \sum_{k=1}^{K} \alpha_k^* R_k u_k^* \right\rangle = 0.$$

This gives us the desired expression for optimal value of TER. It is nonnegative since all terms in its definition (8) are nonnegative. □

Note that the nonnegative value

$$TER(\pi_*) - \langle b, y_* \rangle \quad = \quad \sum_{i=1}^{L} \beta_i^* \tau_i - \sum_{k=1}^{K} \alpha_k^* (\varkappa_k + \langle c_k, u_k^* \rangle) - \langle b, y_* \rangle \tag{14}$$

represents the total rate of accumulation of the capital within the internal market. In general, equilibrium prices, market flows, and participation levels are not unique. Nevertheless, all of them ensure the same value of $TER^* \stackrel{\mathrm{def}}{=} TER(\pi_*)$. We call it the *total excessive revenue* of the market.

Depending on participation level of the agents, we distinguish three categories:

- successful agents with participation level one,
- bankrupted agents with participation level zero,
- marginal agents with participation level between zero and one.

Let us explain the role of marginal agents in the capitalistic economy. First of all, note that marginal agents reach their breaking point, where they make neither a profit nor a loss. For a marginal producer it means that his return is equals to the fixed cost of production. Net saving of a marginal consumer is zero, i.e. his budget is equal to the minimal possible expenditure. Since these agents do not get any positive gain from participation in the market, they can try different levels of market activity. The corresponding strategy of trials and errors finally results in balancing the whole market in the sense of conditions (B1) and (B2). Note that at equilibrium, the consumer spends in average only a fraction of his budget on the market, which covers only a part of his actual needs. The remaining part can be used at alternative markets, which are not included in our model. Such a behavior is typical for poor people, and we can treat the fractional participation coefficient as a measure of *poverty*. Thus, the marginal agents play a crucial role in our approach to market modelling.

Finally, let us describe for our model the structure of the aggregate supply/demand functions. For $\pi = (p, y) \in \mathbb{R}^{n+r}$ we have

$$TER(\pi) = \sum_{k=1}^{K} EPR_k(\pi) + \sum_{i=1}^{L} ECR_i(\pi) + \langle b, y \rangle,$$

where

$$\begin{aligned} EPR_k(\pi) &= (PR_k(\pi))_+ = (PB_k(\pi) - \varkappa_k)_+ \\ &= \max_{u_k \in \mathcal{U}_k} \left(\langle p - c_k - R_k^T y - A_k^T p, u_k \rangle - \varkappa_k \right)_+ , \\ ECR_i(\pi) &= [CR_i(\pi)]_+ = [\tau_i - CE_i(\pi)]_+ \; = \; \max_{v_i \in \mathcal{V}_i} (\tau_i - \langle p, v_i \rangle)_+ \, . \end{aligned}$$

The aggregate supply/demand function is defined by subdifferentials of EPR_k of the producers. It is a multivalued mapping $S : \mathbb{R}^{n+r}_+ \to 2^{\mathbb{R}^{n+r}}$ of the following form:

$$\begin{aligned} S(\pi) \;\; &= \;\; \textstyle\sum_{k=1}^{K} \Big\{ \; \alpha_k (u_k - A_k u_k, -R_k u_k) : \; u_k \in \mathcal{U}_k^*(\pi), \\ & \qquad \alpha_k \in [0,1], \; \alpha_k = \left\{ \begin{array}{ll} 1, & \text{if } PR_k(\pi) > 0, \\ 0, & \text{if } PR_k(\pi) < 0. \end{array} \right\}. \end{aligned} \tag{15}$$

In this definition, u_k is the tentative production volume of k-th producer, α_k is his participation level, and $\alpha_k u_k$ is his real production. Note that the producers also require some resources. This explains the presence of the terms with negative signs in (15).

The aggregate demand function is defined by subdifferentials of ECR_i of consumers, taking with the negative sign. It is a multivalued mapping $D : \mathbb{R}^{n+r}_+ \to 2^{\mathbb{R}^{n+r}}$ defined as

$$D(\pi) \;\; = \;\; \sum_{i=1}^{L} \Big\{ (\beta_i v_i, 0) : \; v_i \in \mathcal{V}_i^*(\pi), \; \beta_i \in [0,1], \; \beta_i = \left\{ \begin{array}{ll} 1, & \text{if } CR_i(\pi) > 0, \\ 0, & \text{if } CR_i(\pi) < 0 \end{array} \right\}. \tag{16}$$

Again, in this definition v_i is the tentative consumption volume of i-th consumer, β_i is his participation level, and $\beta_i v_i$ is the real consumption. Now we can write down expression for the full subdifferential of TER:

$$\partial TER(\pi) \;\; = \;\; S(\pi) - D(\pi) + (0, b)^T, \quad \pi \geq 0. \tag{17}$$

Note that all terms in the definition (16) have the same sign. This is because the current version of our model has only pure consumers. In Section 5, we will allow the consumers to work. Then they become also the producers of the *labor*. In this case, the definition of the demand function must include also the mixed terms.

Note that all functions EPR_k and ECR_i are convex in π. Therefore, in a full correspondence with the common sense, the multivalued supply/demand mapping S is an "increasing function" of price π (a *monotone operator*, e.g. [16]). On the other hand, the multivalued demand mapping D is a "decreasing function" of the price (an *antimonotone operator*). As in the classical microeconomic theory (e.g., [9]), in our model the equilibrium price π_* equalizes the aggregate supply and demand:

$$[S(\pi_*) + (0, b)^T] \ \bigcap \ D(\pi_*) \ \neq \ \emptyset. \tag{18}$$

(For the sake of simplicity we assume here that all equilibrium prices are positive.) In view of Theorem 1, the equilibrium prices are exactly the minimizers of the total excessive revenue TER. Since this function is convex in π, condition (18) is equivalent to the usual first-order optimality condition

$$0 \ \in \ \partial TER(\pi_*). \tag{19}$$

Let us show by a simple example that our concepts lead to intuitively correct equilibrium solutions.

Example 1 *Consider a "market" with single producer and single consumer. Producer is able to produce two different products, containing some useful substance, say sugar, in quantities q_1 and q_2 per unit of weight. Consumer is going to eat at least σ units of sugar per week. Producer controls two factories with maximal production volumes $\bar{u}_1$ and $\bar{u}_2$ per week such that*

$$q_1\bar{u}_1 \ < \ \sigma \ < \ q_1\bar{u}_1 + q_2\bar{u}_2. \tag{20}$$

And let the production cost c_1 and c_2 for these products satisfy inequality $\frac{c_1}{q_1} < \frac{c_2}{q_2}$. Assume that the weekly consumer budget is big enough: $\tau > \frac{c_2}{q_2}\sigma$. What is the equilibrium solution for this market?

In accordance to our model, we define excessive revenue of producer (see (4)) as follows:

$$EPR(p) \ = \max_{\substack{0 \le u_1 \le \bar{u}_1 \\ 0 \le u_2 \le \bar{u}_2}} ((p_1 - c_1)u_1 + (p_2 - c_2)u_2)_+ \ = \ \bar{u}_1 \cdot (p_1 - c_1)_+ + \bar{u}_2 \cdot (p_2 - c_2)_+.$$

At the same time, the excessive revenue of consumer (7) is defined as

$$\begin{aligned} ECR(p) \ &= \max_{v_1, v_2 \ge 0} \left\{ (\tau - v_1 p_1 - v_2 p_2)_+ : \ q_1 v_1 + q_2 v_2 \ge \sigma \right\} \\ &= \ \left(\tau - \sigma \min\left\{ \tfrac{p_1}{q_1}, \tfrac{p_2}{q_2} \right\} \right)_+. \end{aligned}$$

Let us show that the equilibrium prices are defined as

$$p_1^* \ = \ \tfrac{q_1}{q_2} c_2 \ > \ c_1, \quad p_2^* \ = \ c_2.$$

Indeed, in this situation $S(p^*) \ = \ \bar{u}_1 e_1 + [0, \bar{u}_2] e_2$, and $D(p^*) \ = \ \left[\frac{\sigma}{q_1} e_1, \frac{\sigma}{q_2} e_2 \right]$. Hence, condition (18) is satisfied in view of relations (20). The equilibrium price p^* is unique since $0 \in \operatorname{int}(S(p^*) - D(p^*))$. $\square$

5 Simple extension: working customers

The excessive revenue model described in Section 2 can be easily extended in order to capture various additional features of the market. In this section we illustrate this by introducing in our model the *labor market*.

For producers, the production cost includes now the cost of the labor, which is used for computing the salaries of workers. Assume that there exist m different professional skills of the workers. The corresponding vector of their unitary labor prices (e.g., on hourly basis) is $z \in \mathbb{R}^m_+$. For simplicity, we assume that all producers at the market apply the same labor prices. However, these prices are not fixed. They are finally determined by the market conditions (same as the external product prices $y \in \mathbb{R}^r_+$).

The technological labor matrix of k-th producer $L_k \in \mathbb{R}^{m\times n}$ describes the necessary amounts of labor in this particular production environment. Namely, the column $L_k e_j \in \mathbb{R}^m$ represents the amounts of working hours of all professions, which are necessary for producing single units of jth product. Thus, the total remuneration paid by the k-th producer for the production plan $u_k \in \mathcal{U}_k$ is $\langle z, L_k u_k\rangle$.

Hence, for the market with labor, we need to extend our system of prices:

$$\pi \stackrel{\mathrm{def}}{=} (p, y, z) \quad \in \quad \mathbb{R}^{n+r+m}.$$

All definitions are modified now in a natural way:

$$\begin{array}{rcl} PB_k(\pi) & = & \max\limits_{u_k \in \mathcal{U}_k} [\ \langle p - c_k, u_k\rangle - \langle y, R_k u_k\rangle - \langle p, A_k u_k\rangle - \langle z, L_k u_k\rangle \ ]\,, \\[2ex] \mathcal{U}^*_k(\pi) & = & \arg\max\limits_{u_k \in \mathcal{U}_k} [\ \langle p, u_k\rangle - \langle y, R_k u_k\rangle - \langle p, A_k u_k\rangle - \langle z, L_k u_k\rangle \ ]\,, \\[2ex] PR_k(\pi) & = & PB_k(\pi) - \varkappa_k, \\[2ex] EPR_k(\pi) & = & (\ PR_k(\pi)\ )_+\,. \end{array} \tag{21}$$

We assume that each consumer can increase his regular income by accepting a job. For that, i-th consumer has to decide on distribution $t_i \in \mathbb{R}^m_+$ of his total working time $\theta_i \in \mathbb{R}_+$ among m different jobs. Thus, the feasible set for time distribution of ith consumer looks as follows

$$\mathcal{T}_i \quad \stackrel{\mathrm{def}}{=} \quad \left\{ t_i \in \mathbb{R}^m_+ : \sum_{j=1}^m t_i^{(j)} \le \theta_i \right\}.$$

Note that ith consumer may have professional training for several jobs. Denoting by $s_i^{(j)}$ his productivity in performing job j, $j = 1, \dots, m$, (it can happen that some of these values are zeros), we introduce diagonal matrix $S_i = \mathrm{diag}\left(s_i^{(1)}, \dots, s_i^{(m)}\right) \in \mathbb{R}^{m\times m}$. Now, for a feasible time distribution $t_i \in \mathcal{T}_i$, the salary of this consumer can be computed as $\langle z, S_i t_i\rangle$.

In order to compute the optimal time distribution, i-th consumer maximizes his salary. Thus, he solves the problem

$$W_i(z) \stackrel{\mathrm{def}}{=} \max_{t_i \in \mathcal{T}_i} \langle z, S_i t_i\rangle, \quad \mathcal{T}^*_i \stackrel{\mathrm{def}}{=} \mathrm{Arg}\max_{t_i \in \mathcal{T}_i} \langle z, S_i t_i\rangle.$$

This income modifies the revenue of i-th consumer:

$$\begin{array}{rcl} CR_i(\pi) & = & \tau_i + W_i(z) - CE_i(\pi) \\ & = & \tau_i + \max\limits_{t_i \in \mathcal{T}_i} \langle z, S_i t_i \rangle + \max\limits_{v_i \in \mathcal{V}_i} (-\langle p, v_i \rangle). \end{array}$$

As before, his excessive revenue is defined as $ECR_i(\pi) = (CR_i(\pi))_+$.

Similar to the market with goods and resources, we can give now an adapted definition of proper participation levels and real market flows (compare with Definition 1). In the new definition, the time distributions play the same role as the production volumes.

Definition 6 *For a given system of prices and salaries* $\pi = (p, y, z) \in \mathbb{R}^{n+r+m}_+$ *and a system of* tentative market flows

$$F_0 = \left(\left\{ u_k^0 \right\}_{k=1}^K, \left\{ v_i^0 \right\}_{i=1}^L, \left\{ t_i^0 \right\}_{i=1}^L \right) \in \Omega \stackrel{\text{def}}{=} \prod_{k=1}^K \mathcal{U}_k \times \prod_{i=1}^L \mathcal{V}_i \times \prod_{i=1}^L \mathcal{T}_i,$$

the system of participation levels $\gamma = \left(\{\alpha_k\}_{k=1}^K, \{\beta_i\}_{i=1}^L \right) \in [0,1]^{K+L}$ *is called* proper *(with respect to* π *and* F_0*) if it satisfies the following conditions:*

$$\alpha_k = \begin{cases} 1, & \text{if } \langle p - c_k - R_k^T y - A_k^T p - L_k^T z, u_k^0 \rangle > \varkappa_k(\mathcal{U}_k), \\ 0, & \text{if } \langle p - c_k - R_k^T y - A_k^T p - L_k^T z, u_k^0 \rangle < \varkappa_k(\mathcal{U}_k), \end{cases}$$

$$\beta_i = \begin{cases} 1, & \text{if } \langle p, v_i^0 \rangle < \tau_i + \langle z, S_i t_i \rangle, \\ 0, & \text{if } \langle p, v_i^0 \rangle > \tau_i + \langle z, S_i t_i \rangle. \end{cases}$$

Such a triple (π, F_0, γ) *defines a* real market flow $F_r = \left(\left\{ \alpha_k u_k^0 \right\}_{k=1}^K, \left\{ \beta_i v_i^0 \right\}_{i=1}^L, \left\{ \beta_i t_i^0 \right\}_{i=1}^L \right)$. □

Now we can define the equilibrium market flows. Let $F_r = \left(\{\widetilde{u}_k\}_{k=1}^K, \{\widetilde{v}_i\}_{i=1}^L, \left\{ \widetilde{t}_i \right\}_{i=1}^L \right)$ be a real market flow defined by some triple (π, F_0, γ). We need one more balancing condition.

(B3) **The market of labor is balanced.** This means that the amount of labor needed for fulfilling the real production plans does not exceed its available amount. Moreover, the labor markets for jobs with positives salaries ($z^{(j)} > 0$) clear:

$$\sum_{k=1}^K (L_k \widetilde{u}_k)^{(j)} = \sum_{i=1}^L \left(S_i \widetilde{t}_i \right)^{(j)}.$$

The left-hand side of this equation corresponds to the amount of j-th labor required by producers. The right-hand side represents its real amount provided by the workers.

Definition 7 *We say that the set* $\pi \in \mathbb{R}_+^{n+r+m}$ *is the set of* equilibrium prices *if there exists a tentative flow*

$$F^* \in \Omega_*(\pi) \stackrel{\text{def}}{=} \prod_{k=1}^{K} \mathcal{U}_k^*(\pi) \times \prod_{i=1}^{L} \mathcal{V}_i^*(\pi) \times \prod_{i=1}^{L} \mathcal{T}_i^*(\pi)$$

and a proper system of participation levels γ *such that the corresponding real market flow* F_r^* *satisfies the balancing conditions (B1), (B2), and (B3).*

In this case F_r^* *is called the* equilibrium market flow. □

At equilibrium, all producers and consumers tend to maximize their revenues. We define the total excessive revenue of the market with labor by summing up the excessive revenues of all agents:

$$TER(\pi) := \sum_{k=1}^{K} EPR_k(\pi) + \sum_{i=1}^{L} ECR_i(\pi) + \langle y, b \rangle.$$

At the same time, we can prove the statement similar to Theorem 1.

Theorem 5 *The set* $\pi_* = (p_*, y_*, z_*) \in \mathbb{R}_+^{n+r+m}$ *is the set of equilibrium prices if and only if it solves the following minimization problem:* $\min\limits_{\pi \in \mathbb{R}^{n+r+m}} TER(\pi)$.

The proof of this theorem is almost identical to the proof of Theorem 1.

In this section we have shown how we can enrich our model by additional elements, representing some economical relations between different agents. This can be done for many different types of goods trading at the market. The required structural changes are always the same. Namely, some agents introduce a new good with unknown price. Its production needs some expenditures, which reduce the excessive revenues of these agents. However, the new good can be sold at the market and the gain will increase the corresponding revenues. Similarly, we need to modify the excessive revenues of potential consumers of the new good. The equilibrium market price of the new good can be defined by minimizing the total excessive revenue of the market. Note that the whole structure of such market relations can be very complicated since the same agents can be producers of one good and consumers for another one. However, in many situations the existence theorems are trivial since they can be justified by the same arguments as in Theorem 1.

6 Consumption strategies

In the previous sections we assumed that the consumer chooses his regular consumption pattern as a solution of the following optimization problem:

$$\min_{v \in \mathbb{R}_+^n} \left\{ \langle p, v \rangle : \; Qv \geq \sigma \right\}, \tag{22}$$

where p is the vector of product prices. In the real life, dimension of this problem is usually very big. It seems impossible that a normal consumer can approach somehow its solution without being involved in very heavy computations. Fortunately, this impression is wrong. In order to understand this, let us look at the dual form of problem (22).

$$\begin{aligned}
\min_{v\in\mathbb{R}^n_+} \{\langle p, v\rangle : \; Qv \geq \sigma\} \quad &= \quad \min_{v\in\mathbb{R}^n_+} \max_{\lambda\in\mathbb{R}^d_+} \{\langle p, v\rangle + \langle \lambda, \sigma - Qv\rangle\} \\
&= \quad \max_{\lambda\in\mathbb{R}^d_+} \min_{v\in\mathbb{R}^n_+} \left\{\langle \lambda, \sigma\rangle + \langle p - Q^T\lambda, v\rangle\right\} \\
&= \quad \max_{\lambda\in\mathbb{R}^d_+} \left\{\langle \sigma, \lambda\rangle : \; Q^T\lambda \leq p\right\}.
\end{aligned}$$

Thus, the problem dual to (22) is as follows:

$$\max_{\lambda\in\mathbb{R}^d_+} \left\{\langle \sigma, \lambda\rangle : \; \langle q_j, \lambda\rangle \leq p^{(j)}, \; j = 1, \ldots, n\right\}, \tag{23}$$

where $q_j = Qe_j$. In this problem, the elements $\lambda^{(i)}$ can be interpreted as personal prices for corresponding qualities, $i = 1, \ldots, d$. Thus, the value $\langle q_j, \lambda\rangle$ is the personal estimate of the quality of jth product. Let us introduce now the function

$$\psi(\lambda) \quad = \quad \max_{1\leq j\leq n} \tfrac{\langle q_j, \lambda\rangle}{p^{(j)}}. \tag{24}$$

This function selects the products with the best *quality/price ratio*. Of course, the results of selection depend on our current prices for qualities.

Let us rewrite the dual problem (23) using the function ψ.

$$\max_{\lambda\in\mathbb{R}^d_+} \{\langle \sigma, \lambda\rangle : \; \psi(\lambda) \leq 1\} \quad = \quad \max_{\lambda\in\mathbb{R}^d_+} \tfrac{\langle \sigma, \lambda\rangle}{\psi(\lambda)} \quad = \quad \left[\min_{\lambda\in\mathbb{R}^d_+} \{\psi(\lambda) : \; \langle \sigma, \lambda\rangle = 1\}\right]^{-1}.$$

We are going to show that our consumer is able to approach the optimal solution of problem

$$\min_{\lambda} \{\psi(\lambda) : \; \lambda \in \Delta_d(\sigma)\}, \quad \Delta_d(\sigma) \stackrel{\mathrm{def}}{=} \left\{\lambda \in \mathbb{R}^d_+ : \; \langle \sigma, \lambda\rangle = 1\right\}, \tag{25}$$

by implementing the standard consumption cycle (1).

For simplicity, let us assume that for each shopping we have the same budget $h > 0$. During our consumption history, we update the vector of accumulated qualities $s_k \in \mathbb{R}^d$, and the vector of cumulative product consumption $v_k \in \mathbb{R}^n$.

Clearly, $s_0 = 0$ and $\hat{v}_0 = 0$. Before starting the kth shopping, we have already in mind some prices for qualities λ_k. It is convenient to normalize them by inclusion $\lambda_k \in \Delta_d(\sigma)$, which means that the total budget for our life standards is equal to one.

During the kth shopping, we subconsciously determine the set of products with the best quality/price ratio:

$$I_*(\lambda_k) = \left\{j : \; \tfrac{\langle q_j, \lambda_k\rangle}{p^{(j)}} = \psi(\lambda_k)\right\}.$$

If this set contains more than one element, we need to define the sharing vector $x_k \in \Delta_n(e)$, such that $x_k^{(j)} = 0$ for all $j \notin I_*(\lambda_k)$. Then, the budget for buying jth product for

$j \in I_*(\lambda_k)$ becomes $hx_k^{(j)}$, and we buy $\frac{hx_k^{(j)}}{p^{(j)}}$ quantity of this product. Thus, the vector of accumulated qualities is updated as follows:

$$s_{k+1} \;=\; s_k + hg_k, \quad g_k = \sum_{j=1}^{n} \frac{x_k^{(j)}}{p^{(j)}} q_j. \tag{26}$$

It is important, that $g_k \in \partial\psi(\lambda_k)$. Note that $\langle g_k, \lambda_k \rangle = \psi(\lambda_k)$.

The vector of accumulated product consumption is updated as

$$\hat{v}_{k+1}^{(j)} \;=\; \hat{v}_k^{(j)} + h\frac{x_k^{(j)}}{p^{(j)}}, \quad j = 1, \dots, n. \tag{27}$$

Note that these updates maintain the following relations:

$$\hat{v}_k \;\geq\; 0, \quad s_k \;=\; Q\hat{v}_k, \quad k \geq 0. \tag{28}$$

Let us look now at the *gap bound*

$$\delta_k \;=\; h \max_{\lambda \in \Delta_d(\sigma)} \sum_{i=0}^{k} \langle g_i, \lambda_i - \lambda \rangle. \tag{29}$$

Note that

$$\begin{aligned} \delta_k \;&=\; h\sum_{i=0}^{k} \psi(\lambda_i) - \min_{\lambda \in \Delta_d(\sigma)} h \sum_{i=0}^{k} \langle g_i, \lambda \rangle \;=\; h\sum_{i=0}^{k} \psi(\lambda_i) - \min_{\lambda \in \Delta_d(\sigma)} \langle s_{k+1}, \lambda \rangle \\ &=\; h\sum_{i=0}^{k} \psi(\lambda_i) - \min_{1 \leq j \leq d} \frac{s_{k+1}^{(j)}}{\sigma^{(j)}}. \end{aligned}$$

Let us writer down a problem dual to (25)

$$\begin{aligned} \min_{\lambda} \{\psi(\lambda) :\; \lambda \in \Delta_d(\sigma)\} \;&=\; \min_{\lambda \in \mathbb{R}^d_+} \max_{\substack{\mu \in \mathbb{R}, \\ w \in \Delta_n(e)}} \left\{ \sum_{j=1}^{n} w^{(j)} \frac{\langle q_j, \lambda \rangle}{p^{(j)}} + \mu(1 - \langle \sigma, \lambda \rangle) \right\} \\ &=\; \max_{\substack{\mu \in \mathbb{R}, \\ w \in \Delta_n(e)}} \min_{\lambda \in \mathbb{R}^d_+} \left\{ \langle \sum_{j=1}^{n} w^{(j)} \frac{q_j}{p^{(j)}} - \mu\sigma, \lambda \rangle + \mu \right\} \\ (\tilde{v}^{(j)} \stackrel{\mathrm{def}}{=} w^{(j)}/p^{(j)}) \;&=\; \max_{\mu \in \mathbb{R}, \tilde{v}} \{\mu :\; Q\tilde{v} \geq \mu\sigma,\; \tilde{v} \in \Delta_n(p)\} \\ &=\; \max_{s, \tilde{v}} \left\{ \min_{1 \leq j \leq d} \frac{s^{(j)}}{\sigma^{(j)}} :\; s = Q\tilde{v},\; \tilde{v} \in \Delta_n(p) \right\} \end{aligned}$$

Thus, the problem dual to (25) has the following form

$$\max_{s, \tilde{v}} \left\{ \min_{1 \leq j \leq d} \frac{s^{(j)}}{\sigma^{(j)}} :\; s = Q\tilde{v},\; \tilde{v} \in \mathbb{R}^n_+,\; \langle p, \tilde{v} \rangle = 1 \right\}. \tag{30}$$

It is interesting that this problem belongs to the class of models suggested by Lancaster [9] with concave utility function

$$C(v) \;\stackrel{\mathrm{def}}{=}\; \min_{1 \leq j \leq d} \frac{1}{\sigma^{(j)}} \langle Qv, e_j \rangle. \tag{31}$$

However, in our framework this problem can be used only for finding the *product structure* of optimal consumption. The consumption *volumes* must be found from problem (22).

Denote $S_k = (k+1)h$, the total budget for the first $k+1$ shoppings. Then by induction it is easy to see that $\langle p, \hat{v}_{k+1} \rangle \stackrel{(27)}{=} S_k$. Defining now

$$\tilde{\psi}_{k+1} \;=\; \tfrac{1}{k+1} \sum_{i=0}^{k} \psi(\lambda_i), \quad \tilde{s}_{k+1} \;=\; \tfrac{1}{S_k} s_{k+1}, \quad \tilde{v}_{k+1} \;=\; \tfrac{1}{S_k} \hat{v}_{k+1},$$

we can see that $\tilde{v}_{k+1} \in \Delta_n(p)$. Therefore

$$\tfrac{1}{S_k}\delta_k \;=\; \tilde{\psi}_{k+1} - C(\tilde{v}_{k+1}) \;\geq\; 0. \tag{32}$$

Thus, our goal is to present a behavioral strategy for updating the personal prices λ_k, which ensure $\frac{1}{S_k}\delta_k \to 0$ as $k \to \infty$.

For that, we need to introduce new variables $\xi^{(j)} = \frac{1}{\sigma^{(j)}}\lambda^{(j)}$, $j = 1, \dots, d$. If $\lambda \in \Delta_d(\sigma)$, then $\xi \in \Delta_d(e)$. Thus, the new objects have interpretation of probabilities, or *frequencies*.

The value $\xi_k^{(j)}$ has a sense of *relative importance* of quality j for the consumer after k iterations of the consumption process. We start from uniform distribution $\xi_0 = \frac{1}{d}e \in \Delta_d(e)$. For the next iterations, we apply the following rule:

$$\xi_{k+1}^{(j)} \;=\; \frac{\exp\left(-\frac{\tilde{s}_{k+1}^{(j)}}{\mu_{k+1}\sigma^{(j)}}\right)}{\sum\limits_{i=1}^{d} \exp\left(-\frac{\tilde{s}_{k+1}^{(i)}}{\mu_{k+1}\sigma^{(i)}}\right)}, \quad j = 1, \dots, d. \tag{33}$$

This is a standard *Logit* model for detecting the maximal deficiency in the *average level* of consumption of qualities. It has a deviation parameter $\mu_k > 0$ and can be explained as follows.

Between two successive shoppings, we are trying to determine the most deficient quality in our consumption history. For that, we regularly perform subconscious inspections of the average level of their consumption. Since this a subconscious estimating process, its results are random. However, it is natural to assume that

> *the relative importance of each quality approaches the frequency of detecting its average consumption as the lowest one as compared to the standards of life.*

These estimates, divided by the corresponding scale coefficients $\sigma^{(j)}$ become the prices of qualities for the next shopping.

All these computations are done, of course, subconsciously, without any visible efforts from the consumer. Similarly, we assume that our consumer can easily detect products with the best quality-price ratio, which are understood as products with *faire prices*.

It can be proved that with an appropriate strategy of tightening, (e.g. $\mu_k \approx \frac{1}{\sqrt{k+1}}$), the prices for qualities $\lambda_k^{(j)} = \xi_k^{(j)}/\sigma^{(j)}$ defined by (33) converge to the optimal solution of problem (25) (see Appendix). At the same time, the consumption volumes $\tilde{v}_k$ converge to the optimal solution of the dual problem in (30). Thus, our consumer is able to approach the optimal *structure* of the product consumption. Playing now with the budget h, it is possible to ensure the necessary level σ for consumption of qualities.

7 Price dynamics

One of the most important questions within the scope of general equilibrium theory is the description of price dynamics, which should result in equilibrium prices. Usually, for that we need to relate somehow the variation of prices $\pi(t) \geq 0$ in time $t \in \mathbb{R}$ with the excess supply $\bar{z}(t)$, which is the difference between the aggregate supply $\bar{u}(t)$ and demand $\bar{v}(t)$, computed for the current system of prices. One of the most straightforward suggestions is the classical tâtonnement process going back to Walras (e.g., [3]). It consists in postulating the following dynamics

$$\pi'(t) \quad = \quad -\gamma \bar{z}(t), \tag{34}$$

where γ is an appropriate coefficient. However, this very intuitive equation has two shortcomings. Firstly, it is not clear why the entries of the trajectory $\pi(t)$ never become negative. Secondly, the structure of limiting points of $\pi(t)$ can be very complicated. If we assume that the excess supply $\bar{z}(t)$ is computed from Arrow-Debreu model, then $\pi(t)$ can have multiple number of isolated attraction points, [6]. Thus, we have uncertainty in predicting the final state of the price system.

In our model, the excess supply $\bar{z}(t)$ is defined as follows:

$$\bar{z}(t) \quad = \quad \bar{u}(t) - \bar{v}(t) + (0,b)^T, \quad \bar{u}(t) \in S(\pi(t)), \; \bar{v}(t) \in D(\pi(t)) \tag{35}$$

(see (15), (16)). With this definition, equation (34) becomes *potential* since $\bar{z}(t) \in \partial TER(\pi(t))$ (see (17)). Therefore, it can have only a single limiting point which minimizes the total excessive revenue TER, hence, constituting a set of equilibrium prices (see Theorem 1). We prove this statement for a slight modification of the dependence (34), which ensures nonnegativity of prices.

Consider the following dynamics:

$$\frac{d\pi^{(i)}(t)}{dt} \quad = \quad -\pi^{(i)}(t)\, \bar{z}^{(i)}(t)/\gamma_i, \quad i = 1, \ldots, n, \tag{36}$$

where $\bar{z}(t) \in \partial TER(\pi(t))$, and $\gamma_i > 0$ are some scaling parameters. Note that the value $\bar{z}^{(i)}(t)$ corresponds to accumulation rate of unsold volume of product i. Thus, the coefficient γ_i has physical dimension of the volume of this product. It can be interpreted, for example, as stock capacity.

Theorem 6 *Let the optimal set Π_* of optimization problem (9) be bounded. If $\pi(0) > 0$, then $\pi(t) > 0$ for all $t \geq 0$. Moreover, this trajectory converges to a single point from Π^*.*

Proof:
Consider the entropy function $\eta(\tau) = \tau \ln \tau$, $\tau \geq 0$. We can define the Bregman distance between two reals $\tau_1 > 0$ and $\tau_2 \geq 0$ as follows:

$$\begin{aligned} \rho(\tau_1, \tau_2) \quad &= \quad \eta(\tau_2) - \eta(\tau_1) - \eta'(\tau_1)(\tau_2 - \tau_1) \\ &= \quad \tau_2 \ln \tfrac{\tau_2}{\tau_1} - \tau_2 + \tau_1 \; \geq \; 0. \end{aligned}$$

Note that $\rho(\tau_1, 0) = \tau_1$. Therefore, for any sequence of positive numbers $\{\tau_k\}$ and any value $\tau_* \geq 0$, we have $\lim\limits_{k\to\infty} \rho(\tau_k, \tau_*) = 0$ if and only if $\lim\limits_{k\to\infty} \tau_k = \tau_*$. It is important that

$$\rho_1'(\tau_1, \tau_2) \;\; \stackrel{\text{def}}{=} \;\; \tfrac{\partial}{\partial \tau_1} \rho(\tau_1, \tau_2) \; = \; 1 - \tfrac{\tau_2}{\tau_1}.$$

Let us fix an arbitrary $\pi_* \in \Pi_*$, and let $TER_* = TER(\pi_*)$. Consider the function

$$V_{\pi_*}(t) \;=\; \sum_{i=1}^{n} \gamma_i \rho(\pi^{(i)}(t), \pi_*^{(i)}).$$

Since function $TER(\pi)$ is convex and $\bar{z}(t) \in \partial TER(\pi(t))$, we have

$$\begin{aligned} \tfrac{d}{dt} V_{\pi_*}(t) &= \sum_{i=1}^{n} \gamma_i \rho_1'(\pi^{(i)}(t), \pi_*^{(i)}) \tfrac{d\pi^{(i)}(t)}{dt} \stackrel{(36)}{=} -\sum_{i=1}^{n} \left(1 - \tfrac{\pi_*^{(i)}}{\pi^{(i)}(t)}\right) \bar{z}^{(i)}(t)\pi^{(i)}(t) \\ &= -\langle \bar{z}(t), \pi(t) - \pi_* \rangle \;\leq\; -(TER(\pi(t)) - TER_*). \end{aligned}$$

Since Π_* is bounded, this implies that the trajectory $\pi(t)$ converges to Π_*. Denote now by $\pi_* \in \Pi_*$ a limiting point of any convergent sequence $\{\pi(t_k)\}$, $t_k \to \infty$. Since the function $V_{\pi_*}(t)$ decreases monotonically, we conclude that $\lim\limits_{t\to\infty} \pi(t) = \pi_*$. □

Note that the boundedness of the optimal set Π_* is guaranteed by the productivity of the market (see Definition 3 and the proof of Theorem 2).

The equation (36) for price dynamics can be written in the following form:

$$\tfrac{d}{dt}\left(\ln \pi^{(i)}(t)\right) \;=\; -\bar{z}^{(i)}(t)/\gamma_i, \quad i = 1, \dots, n. \tag{37}$$

Hence, we obtain the following discrete-time dynamics:

$$\ln\left(\frac{\pi_{t+1}^{(i)}}{\pi_t^{(i)}}\right) \;=\; -\frac{\bar{z}_t^{(i)}}{\gamma_i}, \quad i = 1, \dots, n. \tag{38}$$

Surprisingly enough, the latter logarithmic dependence may be interpreted as the Weber-Fechner law from psychophysics (e.g., [7]). The Weber-Fechner law describes the relationship between the physical magnitudes of stimuli and their perceived intensity. It states that the subjective sensation is proportional to the logarithm of the stimulus intensity. In our setting, $\frac{-\bar{z}_t^{(i)}}{\gamma_i}$ is the perception resulting from the trade. The expression $\frac{\pi_{t+1}^{(i)}}{\pi_t^{(i)}}$ represents the relation of the previous price stimulus to the next one. This link to psychophysics can open a door for behavioral interpretations of price dynamics.

8 Conclusion

In this paper we presented mathematical models of different market activities using the new convex-concave framework. According to it, the concave variables correspond to agents decisions and convex variables play the role of salaries and prices. Maximization with respect to concave variables represents the rational choices of the agents, and minimization with respect to convex variables leads to the market clearance. Thus, we justify a new price principle: *equilibrium system of salaries and prices corresponds to the minimal value of the total excessive revenue of the market's participants.*

Overall, equilibrium prices and optimal agents' strategies naturally induce the equilibrium market flows, the crucial concept of our paper. The existence and welfare theorems for equilibrium market flow are proven by exploiting its characterization via the convex potential function, namely the total excessive revenue of the market. We argue that this fairly general and novel modelling framework leads to a new understanding of the main principles of functioning of markets of goods, labor, and resources.

In our next paper we will show how to model in a similar way the financial and capital markets, the market of public goods, wholesale markets, etc. Note that all these particular models can be easily incorporated into a general model describing the market of the whole capitalistic economy. Surprisingly enough, our simple model gives a very transparent picture of the economical structure of capitalistic society. It appears that its main feature is the feasibility of poverty for some market participants. By restricted involvement of these marginal (or poor) agents into economic activities, the market clearance is guaranteed. These participant produce (or consume) the amounts of goods which remain available at the market.

From the algorithmic perspective, the new convex-concave framework provides a tractable model of the market. Under tractability we understand the existence of reliable price dynamics whose trajectories converge towards global solutions from any initial price. This price dynamics can be explained by certain numerical schemes aiming at minimizing the total excessive revenue of the market. Note that these schemes are computationally efficient due to the fact that the total excessive revenue is convex with respect to the system of prices and salaries. We are going to discuss the algorithmic details and interpretations in the forthcoming papers.

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Appendix. Primal-dual algorithm of optimal consumption

Let us present now a simple mathematical justification for numerical method for solving a scaled version of problem (25):

$$\min_{\xi \in \Delta_d(e)} \left\{ \phi(\xi) \stackrel{\mathrm{def}}{=} \psi(D^{-1}\xi) \right\},$$

where D is a diagonal matrix with vector σ at its diagonal. Note that $\nabla\phi(\xi) = D^{-1}\nabla\phi(D^{-1}\sigma)$. We denote $\lambda \equiv \lambda(\xi) = D^{-1}\xi$.

Let us introduce the entropy prox-function

$$\eta(\xi) \quad = \quad \sum_{i=1}^{d} \xi^{(i)} \ln \xi^{(i)}, \xi \in \Delta_d(e).$$

Note that this function is strongly convex on $\Delta_d(e)$ with respect to ℓ_1-norm with convexity parameter one.

Let us fix the step parameter $h > 0$. The primal-dual gradient method [15] looks as follows:

$$\xi_0 = \tfrac{1}{d}e, \quad \xi_{k+1} \quad = \quad \arg\min_{\xi \in \Delta_n(e)} \left\{ \sum_{j=0}^{k} \langle \nabla\phi(\xi_j), \xi - \xi_j \rangle + (k+1)\mu_k \eta(\xi) \right\}, \; k \geq 0, \qquad (39)$$

where $\mu_k > 0$ are some scaling coefficients coefficients. Denote $\hat{s}_k = \sum_{j=0}^{k-1} \nabla\phi(\xi_j)$ (thus, $\hat{s}_0 = 0$). Then, in accordance to the rule (39) we have

$$\xi_{k+1}^{(j)} \quad = \quad \frac{\exp\left(-\frac{\hat{s}_{k+1}^{(j)}}{(k+1)\mu_k}\right)}{\sum_{i=1}^{d} \exp\left(-\frac{\hat{s}_{k+1}^{(i)}}{(k+1)\mu_k}\right)}, \quad j = 1, \ldots, d.$$

On the other hand,

$$\hat{s}_{k+1} \quad = \quad \sum_{j=0}^{k} D^{-1}\nabla\psi(\lambda_j),$$

where $\nabla\psi(\lambda_j) = \sum_{i=1}^{n} \frac{x_k^{(i)}}{p^{(i)}} q_i \in \partial\psi(\lambda_k)$, and the sharing vectors $x_k \in \Delta_n(e)$ having elements $x_k^{(i)} = 0$ for $i \notin \left\{ j : \frac{\langle q_j, \lambda_k \rangle}{p^{(j)}} = \psi(\lambda_k) \right\}$. Thus, the process (39) constructs exactly the same sequence of the internal prices for qualities $\{\lambda_k\}$ as the behavioral strategy (26), (33). Moreover, it is easy to see that $hD\hat{s}_k = s_k$.

On the other hand, method (39) is a variant of the method of *dual averaging* (see (2.14) in [15]), with parameters

$$\lambda_k = 1, \quad \beta_{k+1} = (k+1)\mu_{k+1}, \quad k \geq 0.$$

In accordance, to Theorem 1 in [15], we have

$$\hat{\delta}_k \stackrel{\text{def}}{=} \max_{\xi \in \Delta_d} \sum_{i=0}^{k} \langle \nabla\phi(\xi_i), \xi_i - \xi \rangle \quad \leq \quad D(k+1)\mu_{k+1} + \frac{L^2}{2} \left(\frac{1}{\beta_0} + \sum_{i=1}^{k} \frac{1}{i\mu_i} \right),$$

where $L = \max_{\xi \in \Delta_d(e)} \|\nabla\phi(\xi)\|_\infty = \max_{i,j} \frac{1}{\sigma^{(i)}} Q^{(i,j)}$ and $D = \ln d$. Hence,

$$\frac{1}{k+1}\hat{\delta}_k \quad \leq \quad D\mu_{k+1} + \frac{L^2}{2(k+1)} \left(\frac{1}{\beta_0} + \sum_{i=1}^{k} \frac{1}{i\mu_i} \right).$$

Hence, if μ_k gradually goes to zero, then the right-hand side of this inequality vanishes. The best rate of convergence is achieved for $\mu_k \approx O\left(\frac{1}{\sqrt{k+1}}\right)$.

Finally, let us show that the above arguments ensure the convergence of the behavioral strategy (26), (33). Denote $\lambda_k = D^{-1}\xi_k$. Then $\nabla\phi(\xi_k) = D^{-1}\nabla\psi(\lambda_k)$, and we have

$$\begin{aligned} \sum_{i=0}^{k} \langle \nabla\phi(\xi_k), \xi_k - \xi \rangle \quad &= \quad \sum_{i=0}^{k} \langle D^{-1}\nabla\psi(\lambda_k), \xi_k - \xi \rangle \; = \; \sum_{i=0}^{k} \langle \nabla\psi(\lambda_k), D^{-1}\xi_k - D^{-1}\xi \rangle \\ &= \quad \sum_{i=0}^{k} \langle \nabla\psi(\lambda_k), \lambda_k - \lambda \rangle, \quad \xi \in \Delta_d(e), \quad \lambda = D^{-1}\xi \in \Delta_d(\sigma). \end{aligned}$$

Thus, $h\hat{\delta}_{k+1} = \delta_k$. It remains to use equation (32).