Some flavours of topos theory

Francis Borceux

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Francis Borceux
Institut de Recherche en Mathématique et Physique
Université Catholique de Louvain
2 chemin du Cyclotron
1348 Louvain-la-Neuve
Belgium
francis.borceux@uclouvain.be

Introduction

The notion of *sheaf* on a topological space emerged during the period around the second world war, in order to provide an efficient tool to handle local problems, in particular in differential geometry. Sheaves on a topological space admit a straightforward generalization to the case of *locales*: those lattices which mimic the properties of the lattice of open subsets of a space. But the striking generalization has been that of a sheaf on a site, that is, a sheaf on a small category provided with a so-called *Grothendieck topology*. That notion became essential in algebraic geometry, through the consideration of *schemes*. In the late sixties, F.W. Lawvere introduced *elementary toposes*: categories satisfying axiomatically two striking properties typical of the categories of sheaves of sets. Each topos provides a model of intuitionistic logic.

These notes intend to give a quick overview of some relevant aspects of topos theory, without entering the details of the proofs nor the applications in geometry or other fields in mathematics, but giving precise references where to find explicit proofs. Some reasonable familiarity with category theory is assumed.

Lesson 1 focuses on toposes of sheaves: sheaves on a topological space, on a locale and on a site. We prove in particular that these toposes of sheaves satisfy the two characteristic properties which will define elementary toposes. All these toposes are the so-called "Grothendieck toposes".

Lesson 2 switches to *elementary toposes*: those Cartesian closed categories admitting a subobject classifier. We focus on their strong exactness properties which, amazingly, can be inferred from just the two elementary axioms just mentioned. The so-called "axiom of infinity" is also introduced: it allows in particular developing arithmetic, analysis, geometry, and so on, in an elementary topos.

Lesson 3 introduces the notion of internal topology and internal sheaf in an elementary topos, generalizing so various aspects of the theories of sheaves, as in the first lesson. This is also the opportunity to pay attention to Boolean toposes and the law of excluded middle.

Lesson 4 is an essential, but rather technical one: it opens the door to the *internal logic* of a topos. That logic is intuitionistic and allows, in a topos, proving theorems "elementwise" like in the case of sets.

Lesson 5 begins with the study of the "geometric morphisms" of toposes. A special instance of these is the action of a continuous function between the two corresponding toposes of sheaves. Geometric morphisms are central in the study of the so-called "classifying topos" of a given theory: a Grothendieck topos containing a generic model of the theory, a model which allows recapturing all models of that theory in all Grothendieck toposes as images of that generic model along the geometric morphisms.

I thank Marino Gran who invited me to deliver these lessons and, doing so, offered me once more the great pleasure of teaching.

Contents

Lesson 1

Toposes of sheaves

1.1 Sheaves on a topological space

In a first calculus course, one studies in particular the set $\mathcal{C}(\mathbb{R},\mathbb{R})$ of continuous functions on the reals: among these we have x, x^2, $\sin x$, e^x, and so on. But of course, attention is also devoted to functions like $\log x$ and $\sqrt{x}$, which are not defined on the whole of $\mathbb{R}$, but are continuous on their domain of definition: respectively, on $]0,\infty[$ and $[0,\infty[$. Of course, the toplogical definition of a continuous function applies as such to $\sqrt{x}$ when we view it as a function defined on the topological space $[0,\infty[\ \ldots$ but when we view it as a partially defined function on $\mathbb{R}$, the continuity at the point 0 means something different than the continuity at $r > 0$, because $\sqrt{x}$ is not defined on a neighborhood of 0 in $\mathbb{R}$.

The spirit of the sheaf approach is slightly different: at a given point, sheaves focus on those properties which are valid on a neighborhood of that point. For example $\log x$ and $\sqrt{x}$, viewed as functions defined partially on $\mathbb{R}$, exist and are continuous at the neighborhood of each $r > 0$, but not at the neighborhood of 0. Of course, considering open neighborhoods suffices. So our sheaf $\mathcal{C}(-,\mathbb{R})$ of continuous functions on the reals consists in specifying, for each open subset $U \subseteq \mathbb{R}$, the set $\mathcal{C}(U,\mathbb{R})$ of real valued continuous functions on U. Clearly, when $V \subseteq U$ is a smaller open subset, every $f \in \mathcal{C}(U,\mathbb{R})$ restricts as some $f|_V \in \mathcal{C}(V,\mathbb{R})$. Writing $\mathcal{O}(\mathbb{R})$ for the lattice of open subsets of the reals, we get so a contravariant functor

$$\mathcal{C}(-,\mathbb{R})\colon \mathcal{O}(\mathbb{R}) \longrightarrow \mathsf{Set}, \quad U \mapsto \mathcal{C}(U,\mathbb{R})$$

to the category of sets.

Next, given two open subsets U, V and continuous functions $f\colon U \longrightarrow \mathbb{R}$, $g\colon V \longrightarrow \mathbb{R}$ which coincide on $U \cap V$, we can "glue" f and g together to extend them in a continuous function defined on $U \cup V$. And of course, the same process holds when choosing an arbitrary number of continuous functions $f_i\colon U_i \longrightarrow \mathbb{R}$, not just two. So our sheaf of continuous functions satisfies the property:

> Given open subsets $U = \bigcup_{i\in I} U_i$ and continuous functions $f_i \in \mathcal{C}(U_i,\mathbb{R})$
> If for all indices i, j, one has $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$
> Then there exists a unique $f \in \mathcal{C}(U,\mathbb{R})$ such that for all i, $f|_{U_i} = f_i$.

Definition 1.1 Consider a topological space X and its lattice $\mathcal{O}(X)$ of open subsets.
A *presheaf* F on X is a contravariant functor $F\colon \mathcal{O}(X) \longrightarrow \mathsf{Set}$. When $V \subseteq U$ in $\mathcal{O}(X)$, we write

$$F(U) \longrightarrow F(V), \quad a \mapsto a|_V$$

for the action of the functor F on the morphism $V \subseteq U$ of $\mathcal{O}(X)$.
A *sheaf* F on X is a presheaf satisfying the axiom

Given $U = \bigcup_{i\in I} U_i$ in $\mathcal{O}(X)$ and $a_i \in F(U_i)$ for each i
If for all indices i, j one has $a_i|_{U_i\cap U_j} = a_j|_{U_i\cap U_j}$
Then there exists a unique $a \in F(U)$ such that for all i, $a|_{U_i} = a_i$.

The morphisms of presheaves or sheaves are the natural transformations between them. The category of sheaves on a topological space is called a *spatial topos*.

Examples 1.2 The following are examples of sheaves.

1. given a natural number $k \in \mathbb{N}$, the functor

$$\mathcal{O}(\mathbb{R}) \longrightarrow \mathsf{Set}, \quad U \mapsto \mathcal{C}^k(U, \mathbb{R})$$

   mapping an open subset U on the set of k-times differentiable functions from U to $\mathbb{R}$;

2. given topological spaces X and Y, the functor

$$\mathcal{O}(X) \longrightarrow \mathsf{Set}, \quad U \mapsto \mathcal{C}(U, Y)$$

   mapping an open subset U on the set of continuous mappings from U to Y;

3. given a continuous mapping $p\colon Y \longrightarrow X$, the functor

$$\mathcal{O}(X) \longrightarrow \mathsf{Set}, \quad U \mapsto \mathcal{S}(U, Y) = \{s \mid s \in \mathcal{C}(U, Y),\ p \circ s = \mathsf{id}_U\}$$

   mapping an open subset U on the set of continuous sections of p on U. □

Suggestion(s) for further reading

Example 1.2.3 is somehow "generic" as is shown in Sections 2.4, 2.5 and 2.6 of [4]. More precisely, given a sheaf F on a topological space X, consider for every element $x \in X$ the so-called *stalk* of F at x:

$$F_x = \mathsf{colim}_{U \ni x} F(U), \quad U \in \mathcal{O}(X).$$

This is a filtered colimit. Define Y to be the disjoint union of all these stalks and put on Y the final topology for all mappings σ_a^U,

$$\sigma_a^U : U \longrightarrow Y; \quad x \mapsto [a] \in F_x$$

for all $U \in \mathcal{O}(X)$ and $a \in F(U)$. The trivial projection $p\colon Y \longrightarrow X$ is then an étale mapping, that is, a continuous mapping such that for each point $y \in Y$, there exist neighborhoods of y and $p(y)$ on which p restricts as a homeomorphism. The sheaf F we started with is then isomorphic to the sheaf of continuous sections of p. This yields further an equivalence between the category of sheaves on X and that of étale maps over X.

1.2 Sheaves on a locale

Definition 1.1 shows at once that the notion of sheaf on a topological space depends only on the corresponding lattice of subobjects. So one could be tempted to extend this definition to the case of an arbitrary complete lattice: complete, since the definition of sheaf on a topological space refers to coverings $U = \bigcup_{i\in I} U_i$. But completeness does not suffice, because sheaf theory uses also in an essential way the restriction to a smaller open subset $V \subseteq U$. In a topological space, given a covering $U = \bigcup_{i\in I} U_i$ and an open subset $V \subseteq U$, one gets

$$V \cap U = V \cap \bigcup_{i\in I} U_i = \bigcup_{i\in I}(V \cap U_i)$$

so that the covering of U restricts as a covering of $V \cap U = V$. Such a good behavior of coverings is essential for the development of a theory of sheaves.

We shall thus adopt the following definition:

Definition 1.3 A *locale* is a complete lattice in which finite meets distribute over arbitrary joins.

The condition in the definition is thus

$$u \wedge \bigvee_{i\in I} v_i = \bigvee_{i\in I}(u \wedge v_i)$$

for all elements of the locale. Notice that a locale has a top element 1 (the join of all its elements) and a bottom element 0 (the join of the empty family of elements). It has also arbitrary meets (the join of all the lower bounds), but this is of little interest since infima in a locale do not have any relevant property; in the case of topological spaces, such an infimum is the interior of the set theoretical intersection.

Of course we define:

Definition 1.4 Consider a locale L
A *presheaf* on L is a contravariant functor $F\colon L \longrightarrow \mathsf{Set}$. When $v \leq u$ in L, we write

$$F(u) \longrightarrow F(v), \quad a \mapsto a|_v$$

for the action of the functor F on the morphism $v \leq u$ of L.
A *sheaf* on L is a presheaf F satisfying the axiom

> Given $u = \bigvee_{i\in I} u_i$ in L and $a_i \in F(u_i)$ for each i
> If for all indices i, j one has $a_i|_{u_i\wedge u_j} = a_j|_{u_i\wedge u_j}$
> Then there exists a unique $a \in F(u)$ such that for all i, $a|_{u_i} = a_i$.

The morphisms of presheaves or sheaves are the natural transformations between them.
The category of sheaves on a locale is called a *localic topos*.

The families $(a_i)_{i\in I}$ as in Definition 1.4 are generally referred to as *compatible families of elements* along the covering $u = \bigvee_{i\in I} u_i$; $a \in F(u)$ is called the *gluing* of that family. Sheaves on a locale extend thus the theory of sheaves on a topological space.

Let us emphasize the following crucial property:

Theorem 1.5 *Every locale is a Cartesian closed category.*[1]

Sketch of proof Consider three elements u, v, w in a locale L. Put

$$(v \Rightarrow w) = \bigvee\{x \in L \mid x \wedge v \leq w\}.$$

It follows at once that

$$(u \wedge v) \leq w \ \text{ iff } \ u \leq (v \Rightarrow w).$$

But $u \wedge v$ is the product of u and v in the category L. We have thus observed that the functor $(- \wedge v)$ admits $(v \Rightarrow -)$ as a right adjoint. □

[1] A category is Cartesian closed when it has finite products and each functor $- \times B$ admits a right adjoint.

Let us comment the notation $v \Rightarrow w$. Consider a set A provided with the discrete topology. Given V, W in its locale $\wp(A)$ of (open) subsets

$$\begin{aligned}(V \Rightarrow W) &= \{a \in A \big| \{a\} \subseteq (V \Rightarrow W)\} \\ &= \{a \in A \big| \{a\} \cap V \subseteq W\} \\ &= \{a \in A \big| a \in V \text{ implies } a \in W\}\end{aligned}$$

which justifies further the notation.

Corollary 1.6 *Given an element u of a locale L, the element $\neg u = (u \Rightarrow 0)$ is the biggest element whose intersection with u is 0. It is called the pseudo-complement of u.* □

Example 1.7 Every complete Boolean algebra is a locale.

Sketch of proof A Boolean algebra B is in particular a distributive lattice. It follows easily that given three elements u, v, w in B

$$(u \wedge v) \leq w \;\text{ iff }\; u \leq (\complement v \vee w).$$

This shows that the functor $- \wedge v\colon B \longrightarrow B$ admits the right adjoint $\complement v \vee -$, thus preserves all joins. □

So in a complete Boolean algebra, $(v \Rightarrow w) = \complement v \vee w$ and $\neg u = \complement u$.

Suggestion(s) for further reading

In a topological space, an open subset is regular when it is the interior of its closure. The regular open subsets constitute a complete Boolean algebra, where the join of a family is the interior of the closure of the set theoretical union, while a finite meet is the intersection. The "locale" of regular open subsets of a space is generally not isomorphic to a locale of open subsets of a topological space: this is already the case when $X = \mathbb{R}$ (see 1.8.10.d in [3]). Thus localic toposes are more general than spatial toposes.

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1.3 The two basic topos properties

First, a logical peculiarity.

Proposition 1.8 *When F is a sheaf on a locale L, $F(0)$ is a singleton.*

Sketch of proof The bottom element 0 is covered by the empty family of elements of L, thus the empty family of elements glues uniquely in $F(0)$. □

Example 1.9 Each representable functor on a locale L is a sheaf.

Sketch of proof If $u \in L$, the corresponding representable functor has value the singleton on each $v \leq u$, and the empty set elsewhere. □

Let us now focus on the two most characteristic properties of a topos. As already recalled in Theorem 1.5, a category $\mathcal{E}$ with finite products is *Cartesian closed* when each functor $- \times B$ admits a right adjoint $(-)^B$, yielding thus

$$\mathcal{E}(A \times B, C) \cong \mathcal{E}(A, C^B).$$

When $\mathcal{E} = \mathsf{Set}$, $C^B = \mathsf{Set}(B, C)$ is the set of mappings from B to C and two mappings

$$f\colon A \times B \longrightarrow C, \quad g\colon A \longrightarrow \mathsf{Set}(B, C)$$

correspond to each other via the formula $f(a, b) = g(a)(b)$.

Theorem 1.10 *The topos of sheaves on a locale L is Cartesian closed.*

Sketch of proof Given two sheaves G and H, and an element $u \in L$, define

$$H^G(u) = \mathsf{Nat}(G|_u, H|_u)$$

where $G|_u$ and $H|_u$ are the restrictions of G and H at the level u, that is, coincide with G and H on each $v \leq u$, and take empty values elsewhere. This implies at once

$$\mathsf{Nat}(F \times G, H) \cong \mathsf{Nat}(F, H^G)$$

for all sheaves F. (See Theorem 2.3.4 in [4]) □

The second basic property of a topos refers to the following notion:

Definition 1.11 By a subobject classifier in an arbitrary category is meant a monomorphism $t\colon \mathbf{1} \rightarrowtail \Omega$, with $\mathbf{1}$ a terminal object, so that for each object A, there is a bijection between the subobjects $s\colon S \rightarrowtail A$ and the morphisms $\varphi_S\colon A \longrightarrow \Omega$, this bijection giving rise to pullbacks

$$\begin{array}{ccc} S & \longrightarrow & \mathbf{1} \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ A & \xrightarrow[\varphi_S]{} & \Omega \end{array}$$

The arrow φ_S is called the *characteristic morphism* of the subobject $s\colon S \rightarrowtail A$.

This definition refers to subobjects (i.e. isomorphism classes of monomorphisms) and not to individual monomorphisms, because a pullback is defined only up to isomorphism.

In the category of sets, the subobject classifier is

$$\{1\} \rightarrowtail \{0, 1\}$$

and the chararacteristic mapping of a subset $S \subseteq A$ is, as expected

$$\varphi(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \notin S \end{cases}$$

Writing true instead of 1 and false instead of 0, the subobject classifier becomes

$$t\colon \{\mathsf{true}\} \rightarrowtail \{\mathsf{false}, \mathsf{true}\}$$

and the characteristic mapping

$$\varphi(a) = \begin{cases} \mathsf{true} & \text{if } a \in S \\ \mathsf{false} & \text{if } a \notin S \end{cases}$$

This characteristic mapping is thus the "truth table" of the formula $a \in S$ when you view the Ω-object in Set as the "object of truth values".

Theorem 1.12 *Let L be a locale. For every element $u \in L$, putting*

$$\Omega(u) = \downarrow u = \{v | v \in L,\ v \leq u\}.$$

defines a sheaf on L which, together with the morphism

$$t\colon \mathbf{1} \longrightarrow \Omega, \quad t_u(\star) = u$$

is a subobject classifier.

Sketch of proof The morphism t is of course a monomorphism, because $\mathbf{1}$ is the terminal object; $\mathbf{1}(u)$ is the singleton for each $u \in L$. Given a subsheaf $S \subseteq F$, one has $S(u) \subseteq F(u)$ for each $u \in L$. Given $a \in F(u)$

$$\varphi_u(a) = \bigvee \{v | a|_v \in S(v)\}$$

yields the expected result with thus, by the sheaf condition, $\varphi_u(a)$ the biggest element $w \in L$ such that $a|_w \in S(w)$. (See Theorem 2.3.2 in [4].) □

With in mind the case of sets, think now Ω as the sheaf of truth values of our "logic of sheaves": $\varphi_u(a)$ is then worth being called the "truth value" of the formula $a \in S$, the biggest level were this formula becomes true.

Let us take an example. In the topos of sheaves on the reals, consider the inclusion $\mathcal{C}^1(-,\mathbb{R}) \rightarrowtail \mathcal{C}(-,\mathbb{R})$ of the sheaf of differentiable functions in the sheaf of continuous functions (see Examples 1.2). When you ask your students: *Is the "absolute value function" $|x|$ differentiable?* ... what do you prefer as an answer: "false" or "true *on* $\mathbb{R} \setminus \{0\}$"? The first answer is the correct one in classical logic, the second answer is the correct one in the logic of sheaves, as we shall see when studying the internal logic of toposes.

Corollary 1.13 *In the topos of sheaves on a locale L, for every element $u \in L$ and every sheaf G, the set $\Omega^G(u)$ is isomorphic to the set of subsheaves of $G|_u$.*

Sketch of proof By Theorems 1.10 and 1.12, since a morphism $G|_u \longrightarrow \Omega|_u$ extends at once as a morphism $G|_u \longrightarrow \Omega$. □

Thus again, we shall think Ω^G as the "sheaf of subobjects of G".

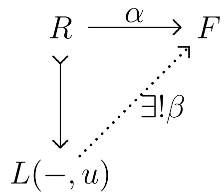
1.4 Sheaves on a site

Let us first revisit the notion of sheaf on a locale L in more categorical terms. Let F be a presheaf on L. Given a covering $u = \bigvee_{i\in I} u_i$, a family $\big(a_i \in F(u_i)\big)_{i\in I}$ is compatible precisely when you can extend it as a compatible family along all the elements $v \in L$ smaller than some u_i. It would thus have been equivalent to express the sheaf condition in terms of only "downward directed" coverings.

But when working with a downward directed covering $u = \bigvee_{i\in I} u_i$ and putting

$$R(u_i) = \{\star\} \ \text{ and } \ R(v) = \emptyset \text{ otherwise}$$

we get now a subpresheaf $R \rightarrowtail L(-,u)$. Moreover giving a compatible family $\big(a_i \in F(u_i)\big)_{i\in I}$ in a sheaf F is now the same as giving a natural transformation $\alpha\colon R \longrightarrow F$. Next, saying that the compatible family glues as some element $a \in F(u)$ is equivalent to saying that the natural transformation β, corresponding to a by the Yoneda lemma, makes the following triangle commutative:



So, calling *covering* a subobject $R \rightarrowtail L(-,u)$ such that

$$u = \bigvee \{v | R(v) \neq \emptyset\}$$

the sheaf condition on a presheaf F is the so-called *orthogonality* condition to all the covering subobjects.[2]

One could be tempted to define a site as a small category $\mathcal{C}$ provided, for each object $C \in \mathcal{C}$, with a family of subobjects of $\mathcal{C}(-,C)$ chosen as the "covering ones". But if one expects to extend elegantly the properties encountered in the case of a locale, the families of covering subobjects should mimic the properties of the covering families in a locale L:

1. for each $u \in L$, u covers u.

2. if the u_i's cover u and $v \leq u$, then the $v \wedge u_i$'s cover v.

3. let $v_j \leq u$ be an arbitrary family; if the u_i's cover u and for each i, the $v_j \wedge u_i$'s cover u_i, then the v_j's cover u.

Definition 1.14 Let $\mathcal{C}$ be a small category. Call a subobject of a representable functor $\mathcal{C}(-,C)$ a *sieve* on C. A *Grothendieck topology* $\mathcal{T}$ on $\mathcal{C}$ consists in specifying, for each object $C \in \mathcal{C}$, a family $\mathcal{T}(C)$ of sieves on C (called the *covering sieves*), so that the following axioms are satisfied:

1. for each C, $\mathcal{C}(-,C)$ covers C.

2. if R covers C and $f\colon D \longrightarrow C$, then $\mathcal{C}(-,f)^{-1}(R)$ covers D.

3. let S be an arbitrary sieve on C; if a sieve R covers C and for every $D \in \mathcal{C}$, $f \in R(D)$, $\mathcal{C}(-,f)^{-1}(S)$ covers D, then S covers D.

A small category provided with a Grothendieck topology is called a *site*.

Definition 1.15 Let $(\mathcal{C}, \mathcal{T})$ be a site.
A *presheaf* on $(\mathcal{C}, \mathcal{T})$ is a contravariant functor $\mathcal{C} \longrightarrow \mathsf{Set}$.
A *sheaf* on $(\mathcal{C}, \mathcal{T})$ is a presheaf F, orthogonal to every covering sieve.
A morphism of sheaves or presheaves is a natural transformation between them.
The category of sheaves on a site is called a *Grothendieck topos*.

Examples 1.16 The following are examples of Grothendieck toposes:

1. every localic topos;

2. the category of sets;

3. the category of presheaves on a small category $\mathcal{C}$;

[2]In a category, an object X is said *orthogonal* to a morphism $f\colon A \longrightarrow B$ when each morphism $A \longrightarrow X$ factors uniquely through f.

4. the category of G-sets for a group G;

5. the terminal category.

Sketch of proof Everything has been done to recapture the localic toposes as Grothendieck ones. And the category of sets is that of sheaves on the singleton. Presheaves are just sheaves for the topology having $\mathcal{C}(-, C)$ as only sieve covering C. And G-sets are just presheaves on G viewed as a one point category. Finally, declaring covering all the subobjects of $\mathcal{C}(-, C)$, the only possible sheaf is the constant functor on the singleton, because the empty sieve is covering. □

Suggestion(s) for further reading

In Example 1.16.5, the representable functors are not sheaves, except when $\mathcal{C}$ is equivalent to the terminal category. In the other examples, the representable functors are sheaves

The Grothendieck topologies on a small category, ordered by inclusion, constitute a locale (see Proposition 3.2.13 in [4]). As a consequence, given a small category $\mathcal{C}$, there exists a biggest Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$ such that all representable functors are sheaves. It is called the *canonical topology* on $\mathcal{C}$. (See Proposition 3.2.13 in [4].)

1.5 The associated sheaf functor

Given a site $(\mathcal{C}, \mathcal{T})$, this section investigates the existence of the sheaf aF universally associated with a presheaf F. That is, we want to prove the existence of a left adjoint to the inclusion $\mathsf{Sh}(\mathcal{C}, \mathcal{T}) \subseteq \mathsf{Pr}(\mathcal{C})$ of the category of sheaves in that of presheaves.

Assuming that the problem is solved, given a covering sieve $R \rightarrowtail \mathcal{C}(-, C)$ and a morphism $f\colon R \longrightarrow F$, by the sheaf condition we get a unique factorization g as in the diagram

$$\begin{array}{ccc} R & \rightarrowtail & \mathcal{C}(-, C) \\ {\scriptstyle f}\downarrow & & \vdots\, {\scriptstyle g} \\ F & \xrightarrow[\eta_F]{} & a(F) \end{array}$$

where η_F is the unit of the adjunction. By the Yoneda lemma, giving g is giving an element of $a(F)(C)$: thus each f as above must yield an element of $a(F)(C)$. This explains why we are interested in the following construction.

Proposition 1.17 *Consider a site $(\mathcal{C}, \mathcal{T})$. For every presheaf F and every object $C \in \mathcal{C}$, define*

$$\alpha(F)(C) = \mathsf{colim}_{R \in \mathcal{T}(C)} \mathsf{Nat}(R, F).$$

This extends at once as a presheaf $\alpha(F)$ and further, as a left exact functor

$$\alpha\colon \mathsf{Pr}(\mathcal{C}) \longrightarrow \mathsf{Pr}(\mathcal{C})$$

on the category of presheaves on $\mathcal{C}$.

Sketch of proof Given a site $(\mathcal{C}, \mathcal{T})$, the intersection of two covering sieves on an object C is easily seen to be still a covering sieve. Therefore the colimit in the definition of α is filtered. The result follows then from the commutation of finite limits with filtered colimits in Set, thus also in every category of presheaves. See Section 3.3 in [4] for more details. □

The considerations which led to the definition of α could give hope that $\alpha(F)$ is the corresponding associated sheaf ... but this is not the case. Given a covering sieve R, the colimit process has introduced morphisms $R \longrightarrow \alpha(F)$ which do not arise from morphisms $R \longrightarrow F$, and thus do not necessarily extend. So the idea is to repeat the operation iteratively, hoping that at some stage, it will stop ... and in fact, we shall not have to wait for long!

To clarify the language, let us introduce an intermediate notion: in Definition 1.15, we keep only the uniqueness condition in the orthogonality condition.

Definition 1.18 Let $(\mathcal{C}, \mathcal{T})$ be a site. A *separated presheaf* on this site is a presheaf F such that, given a covering sieve $r\colon R \rightarrowtail \mathcal{C}(-, C)$ and a morphism $f\colon R \longrightarrow F$, there is at most one factorization of f through r.

Proposition 1.19 *In the situation of Proposition 1.17:*

1. *the presheaf $\alpha(F)$ is separated;*

2. *when F is separated, $\alpha(F)$ is a sheaf.*

Sketch of proof See Section 3.3 of [4] for the details of the proof. □

We obtain so the expected result:

Theorem 1.20 *Let $(\mathcal{C}, \mathcal{T})$ be a site. The category $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ is a full reflexive subcategory of the category $\mathsf{Pr}(\mathcal{C})$ of presheaves. The reflection, called the* associated sheaf functor*, is the left exact functor $a = \alpha\alpha$.*

Sketch of proof See Theorem 3.3.12 in [4]. □

1.6 Exactness properties of Grothendieck toposes

First of all:

Proposition 1.21 *A Grothendieck topos is complete and cocomplete.*

Sketch of proof Given a site $(\mathcal{C}, \mathcal{T})$, the category $\mathsf{Pr}(\mathcal{C})$ is complete and cocomplete: limits and colimits are computed pointwise. The category $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ is complete and cocomplete as a full reflexive subcategory of $\mathsf{Pr}(\mathcal{C})$. □

Proposition 1.22 *In a Grothendieck topos $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$*

1. *finite limits commute with filtered colimits;*[3]

2. *colimits are universal;*[4]

3. *sums are disjoint.*[5]

Sketch of proof All these properties hold in Set, thus in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves where limits and colimits are computed pointwise.

By Theorem 1.20, the category $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ is stable in $\mathsf{Pr}(\mathcal{C})$ under limits, while colimits are obtained by applying the associated sheaf functor to the corresponding colimit computed in $\mathsf{Pr}(\mathcal{C})$. This allows to conclude, since the associated sheaf functor preserves all colimits and finite limits (thus also in particular, monomorphisms and the initial object). See Section 3.4 in [4] for more details and properties. □

[3] A diagram is filtered when it contains a cocone on each finite subdiagram.
[4] A colimit is universal when it is preserved by pulling back.
[5] $A_1 \amalg A_2$ is disjoint when the $s_i\colon A_i \longrightarrow A_1 \amalg A_2$ are monomorphisms and their intersection is 0.

Proposition 1.23 *Every Grothendieck topos* $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ *is a regular and exact category*[6].

Sketch of proof By Proposition 1.22.2, coequalizers are preserved by pulling back, thus the category $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ is regular. To prove its exactness, let R be an equivalence relation on A in $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$. Consider its quotient q in the category $\Pr(\mathcal{C})$ of presheaves.

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \xrightarrow{\;q\;} A/R.$$

This is both a coequalizer and a kernel pair in the category of presheaves, because $\Pr(\mathcal{C})$ is exact, since that property holds pointwise in Set. Applying the associated sheaf functor, which preserves coequalizers and kernel pairs, yields the expected result. □

1.7 Back to the basic properties

In this section, we want to focus on the two properties, already studied in Section 1.3 for localic toposes, and which will be taken as axioms for an elementary topos: Cartesian closedness and the existence of a subobject classifier.

Proposition 1.24 *Every Grothendieck topos is Cartesian closed.*

Sketch of proof Let $(\mathcal{C}, \mathcal{T})$ be a site. The Cartesian closedness means the existence, given two sheaves G and H, of a sheaf H^G such that for every sheaf F

$$\mathsf{Nat}(F \times G, H) \cong \mathsf{Nat}(F, H^G).$$

Let us begin with the case of the topos of presheaves (see Example 1.16.3). If Cartesian closedness holds, putting $F = \mathcal{C}(-, C)$, the Yoneda lemma indicates that necessarily,

$$H^G(C) \cong \mathsf{Nat}\big(\mathcal{C}(-, C), H^G\big) \cong \mathsf{Nat}\big(\mathcal{C}(-, C) \times G, H\big).$$

Defining thus

$$H^G(C) = \mathsf{Nat}\big(\mathcal{C}(-, C) \times G, H\big)$$

and writing F as a colimit of representable functors[7], yields the expected result in the topos of presheaves.

To conclude in the case of sheaves, it suffices to observe that when G and H are sheaves, so is H^G. In fact, for H^G being a sheaf, it suffices that H be a sheaf (see 3.4.17 in [4]). □

Proposition 1.25 *Every Grothendieck topos has a subobject classifier.*

[6]An epimorphism is regular when it is a coequalizer. A finitely complete category $\mathcal{C}$ is regular when regular epimorphisms are pullback stable. A subobject $R \leq A \times A$ is an equivalence relation when (f, g) factoring through R is an equivalence relation on each set $\mathcal{C}(X, A)$ of arrows. In a regular category, every equivalence relation admits a quotient and every morphism has an image, that is, factors as a regular epimorphism followed by a monomorphism. The equivalence relation R is effective when it is the kernel pair of its quotient, i.e. the pullback of the quotient morphism with itself. The regular category is exact when every equivalence relation is effective.

[7]Every Set-valued functor is the colimit of the diagram of representable functors over it.

Sketch of proof Let $(\mathcal{C}, \mathcal{T})$ be a site. Let us again begin with the case of the topos of presheaves on $\mathcal{C}$. By the Yoneda lemma and the definition of a subobject classifier, we must define the Ω-object of presheaves as

$$\Omega(C) \cong \mathsf{Nat}\big(\mathcal{C}(-,C),\Omega\big) \cong \big\{R \big| R \subseteq \mathcal{C}(-,C)\big\}.$$

The characteristic morphism $\varphi\colon F \longrightarrow \Omega$ of a subpresheaf $S \rightarrowtail F$ is given by, for every $C \in \mathcal{C}$

$$\varphi_C\colon F(C) \longrightarrow \Omega(C), \quad a \mapsto \big(\varphi_C(a) \rightarrowtail \mathcal{C}(-,C)\big)$$

where

$$\varphi_C(a)(D) = \big\{f\colon D \to C \big| F(f)(a) \in S(D)\big\}.$$

In the case of the topos of sheaves on $\mathcal{C}, \mathcal{T})$, using further the adjunction involving the associated sheaf functor, one must now have, with Ω the subobject classifier of sheaves:

$$\Omega(C) \cong \mathsf{Nat}\big(\mathcal{C}(-,C),\Omega\big) \cong \mathsf{Nat}\big(a\mathcal{C}(-,C),\Omega\big) \cong \{S \mid S \subseteq a\mathcal{C}(-,C) \text{ sub-sheaf}\}$$

This yields again the expected result (see Example 5.2.9 in [4]). An alternative description of the object Ω for sheaves will be given in Theorem 3.12. □

Corollary 1.26 *Given a group G, the subobject classifier of the topos of G-sets is the two -point set $\{\emptyset, G\}$.*

Sketch of proof By Theorem 1.25, since by the existence of inverses in G, if a subobject of the representable G-set G contains an element g, it contains the whole of G. □

Let us conclude this lesson with exhibiting another link between Grothendieck toposes and locales.

Proposition 1.27 *In a Grothendieck topos, the subobjects of every object constitute a locale.*

Sketch of proof This is a property involving only finite intersections and arbitrary unions of subobjects, together with the commutation property between these. The result holds in Set, thus holds pointwise in every topos of presheaves, thus holds in every topos of sheaves, since the associated sheaf functor preserves all these ingredients. □

Lesson 2

Elementary toposes

2.1 The topos axioms

As we shall observe, amazingly enough, all main characteristic properties of Grothendieck toposes can be inferred from the two basic properties of toposes, already mentioned in Sections 1.3 and 1.7.

Definition 2.1 An *elementary topos* is a category $\mathcal{E}$ satisfying the following two axioms:

1. $\mathcal{E}$ is Cartesian closed;
2. $\mathcal{E}$ admits a subobject classifier.

In a topos $\mathcal{E}$, we shall write $(-)^B$ for the right adjoint to the functor $- \times B$, yielding thus the isomorphisms

$$\mathcal{E}(A \times B, C) \cong \mathcal{E}(A, C^B)$$

and we shall write $t\colon \mathbf{1} \rightarrowtail \Omega$ for the subobject classifier. C^B is trivially functorial in C and B, covariantly in C and contravariantly in B.

Examples 2.2 Here are examples of elementary toposes:

1. every Grothendieck topos, thus in particular every localic topos, every category of G-sets for a group G, the category of sets, and so on;
2. the category of finite sets and, more generally, the category of presheaves of finite sets on a finite category and the category of sheaves of finite sets on a finite site.

In the case of a localic topos (see Corollary 1.13), we have observed that Ω^A is the "sheaf of subobjects of A". In an elementary topos, let us at least observe that the "global elements" of Ω^A, that is the morphisms $\mathbf{1} \longrightarrow \Omega^A$, are by Cartesian closedness in bijection with the morphisms $A \cong \mathbf{1} \times A \longrightarrow \Omega$, that is, by the subobject classifier axiom, with the subobjects of A. Therefore, also in the case of an elementary topos, we should intuitively think Ω^A as being the "object of subobjects of A".

Suggestion(s) for further reading

When G is a group and B, C are G-sets, the G-set C^B is isomorphic to the set of ordinary mappings from B to C, provided with a G-action. Together with Corollary 1.26, this shows that the category of finite G-sets, for an arbitrary group G, is an elementary topos.

2.2 Some set theoretical notions in a topos

Let us observe at once that various usual set-theoretical notions translate in an elementary topos.

Definition 2.3 Given an object A of an elementary topos $\mathcal{E}$, the characteristic morphism of the diagonal of A is called the *equality* on A.

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{1} \\ {\scriptstyle \Delta_A}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ A \times A & \xrightarrow[=_A]{} & \Omega \end{array}$$

It will be just written $=$ when no confusion can occur.

In the case of a localic topos, we have thus, for $u \in L$ and $(a, b) \in A(u) \times A(u)$

$$=_u (a, b) = \bigvee \{ v \leq u \mid a|_v = b|_v \}$$

that is, $a =_u b$ is the highest level where the restrictions of a and b become equal ... thus intuitively in the spirit of the logic of sheaves, the "truth value" of the formula $a = b$.

Definition 2.4 Given an object A of an elementary topos $\mathcal{E}$, the morphism $=_A$ of Definition 2.3 corresponds by Cartesian closedness to a morphism

$$\{\cdot\}_A \colon A \longrightarrow \Omega^A$$

which is called the *singleton* on A, and just written $\{\cdot\}$ when no confusion can occur.

In the case of a localic topos, given an element $a \in A(u)$, the morphism $\{\cdot\}_u$ maps a on the natural transformation $A|_u \Rightarrow \Omega|_u$ corresponding to the subsheaf of $A|_u$, which has precisely value $\{a|_v\}$ on each $v \leq u$.

Definition 2.5 Given an object A of an elementary topos $\mathcal{E}$, the identity on Ω^A corresponds by Cartesian closedness to a morphism

$$\in_A \colon A \times \Omega^A \longrightarrow \Omega$$

which is called the *membership relation* on A, and just written $\in$ when no confusion can occur.

Again in the case of a localic topos, given $(a, S) \in A(u) \times \Omega^A(u)$, thus $S \subseteq A|_u$,

$$\in_u (a, S) = \bigvee \{ v \leq u \mid a|_v \in S(v) \}.$$

which is thus the highest level where the restriction of a lies in S ... so again, intuitively in the spirit of the logic of sheaves, the truth value of the formula $a \in S$.

Those considerations allow to prove:

Proposition 2.6 *A topos has finite limits.*

Sketch of proof By assumption, a topos has binary products and a terminal object. The equalizer of two morphisms $(f, g)\colon A \rightrightarrows B$ is the subobject of A classified by the morphism

$$A \xrightarrow{(f,g)} B \times B \xrightarrow{=_B} \Omega. \qquad \square$$

Lawvere's original definition of an elementary topos required also the existence of finite colimits. It has been later observed that this axiom is redundant, but all proofs of this fact are quite involved (see Section 5.7 in [4]) and use in particular various exactness results presented in our next sections.

Theorem 2.7 *An elementary topos has finite colimits.*

Sketch of proof A first technical proof has been presented by C.J. Mikkelsen at an Oberwolfach meeting in 1972. Bob Paré's more conceptual proof uses instead the Beck monadicity criterion (see Theorem 4.4.4 in [3]) to prove that the functor

$$\Omega^{(-)}\colon \mathcal{E}^{op} \longrightarrow \mathcal{E}$$

is monadic.[1] When this is done, since $\mathcal{E}$ has finite limits, so does the monadic category $\mathcal{E}^{op}$ over it (see Proposition 4.3.1 in [3]); thus $\mathcal{E}$ has finite colimits.

We refer to Theorem 5.7.3 in [4] for the details of the proof. Three conditions must be satisfied.

1. $\Omega^{(-)}$ *must have a left adjoint.* The isomorphisms
$$\mathcal{E}^{op}(\Omega^A, B) \cong \mathcal{E}(B, \Omega^A) \cong (B \times A, \Omega) \cong \mathcal{E}(A, \Omega^B)$$
show that $\Omega^{(-)}$ is its own left adjoint.

2. $\Omega^{(-)}$ *must reflect isomorphisms.* Given $f\colon A \longrightarrow B$ such that $\Omega^f\colon \Omega^B \longrightarrow \Omega^A$ is an isomorphism, one proves that Ω^f restricts as a morphism at the level of " singletons",
$$\begin{array}{ccc} \Omega^B & \xrightarrow{\Omega^f} & \Omega^A \\ {\scriptstyle \{\cdot\}}\uparrow & & \uparrow{\scriptstyle \{\cdot\}} \\ B & \dashrightarrow & A \end{array}$$
and that factorization is the inverse of f;

3. $\mathcal{E}^{op}$ *has some specified coequalizers which are preserved by* $\Omega^{(-)}$. But $\mathcal{E}^{op}$ has all coequalizers, since $\mathcal{E}$ is finitely complete by Proposition 2.6; the rest is then an exactness property in $\mathcal{E}$ relating equalizers and the "objects of subobjects". $\square$

This result is reminiscent of the set-theoretical result stating that complete atomic Boolean algebras are monadic over sets and constitute a category equivalent to the dual of sets.

[1] A monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \varepsilon, \mu)$ where T is an endofunctor on $\mathcal{C}$ and $\varepsilon\colon \mathsf{id}_{\mathcal{C}} \Rightarrow T$, $\mu\colon T \circ T \Rightarrow T$ are natural transformations satisfying axioms which mimic those for being a monoid. A $\mathbb{T}$-algebra is a pair (C, ξ) where $\xi\colon T(C) \to C$ satisfies suitable axioms with respect to ε and μ. The category $\mathcal{C}^{\mathbb{T}}$ of $\mathbb{T}$-algebras and the corresponding forgetful functor $\mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ are said to be monadic over $\mathcal{C}$.

2.3 The slice toposes

This section could appear as just providing other examples of toposes. In fact, the results which follow are essential in the development of topos theory; in particular, they will play a central role in developing the internal logic of a topos. Section 5.8 of [4] contains detailed proofs of the various results of this section.

Proposition 2.8 *Given an elementary topos $\mathcal{E}$ and an object $I \in \mathcal{E}$, the slice category $\mathcal{E}/I$ is still a topos.*

Sketch of proof The proof is quite involved, but its spirit is easily grasped in the case of the topos of sets.

Given a set I, the slice category Set/I of arrows over I can be equivalently seen as the category of I-families of sets: an arrow $p\colon A \longrightarrow I$ yields the family $\left(p^{-1}(i)\right)_{i\in I}$ and conversely, a family $(A_i)_{i\in I}$ of sets yields the set $A = \coprod_{i\in I} A_i$, with the obvious projection $p\colon A \longrightarrow I$ mapping the whole of A_i on $i \in I$. The slice category Set/I is thus equivalent to the category of I-families of sets, that is, to the power category Set^I. In that particular case of sets, this is trivially a topos, with the exponentiation and the subobject classifier defined pointwise as in Set, for every index $i \in I$.

In the case of sets, viewed as a mapping $X \longrightarrow I$, the constant I-family $(\Omega)_{i\in I}$ is simply $p_I\colon I \times \Omega \longrightarrow I$. Let us write Ω_I to indicate this object $p_I\colon I \times \Omega \longrightarrow I$ of $\mathcal{E}/I$, in an arbitrary topos $\mathcal{E}$. The terminal object $\mathbf{1}_I$ of $\mathcal{E}/I$ is the identity on I and the subobject classifier $t_I\colon \mathbf{1}_I \longrightarrow \Omega_I$ of $\mathcal{E}/I$ is the the monomorphism

$$t_I\colon I \xrightarrow{\;\cong\;} I \times \mathbf{1} \xrightarrow{\mathsf{id}_I \times t} I \times \Omega.$$

The explicit description of the exponentiation in $\mathcal{E}/I$ is more involved (see Theorem 5.8.1 in [4]). Our Corollary 2.10 will provide an interesting description of it. □

Theorem 2.9 *Consider a morphism $f\colon I \longrightarrow J$ in an elementary topos $\mathcal{E}$. The pullback functor*

$$f^{-1}\colon \mathcal{E}/J \longrightarrow \mathcal{E}/I$$

preserves exponentiation and the subobject classifier. Moreover, it admits both a left adjoint Σ_f and a right adjoint π_f, thus preserves in particular all existing limits and colimits.

Sketch of proof The existence of the left adjoint Σ_f is a general fact which has nothing to do with the topos structure: $\Sigma_f(p) = f \circ p$. The existence of the right adjoint π_f is a much deeper result. With in mind Proposition 2.8 and using the notation in its proof, consider the following commutative rectangle:

$$\begin{array}{ccc} X & & \\ {\scriptstyle p}\downarrow & & \\ I & \xrightarrow[f]{} & J \end{array} \qquad\qquad \begin{array}{ccc} (\mathcal{E}/J)^{op} & \xrightarrow{(f^{-1})^{op}} & (\mathcal{E}/I)^{op} \\ {\scriptstyle \Omega_J^{(-)}}\downarrow & & \downarrow{\scriptstyle \Omega_I^{(-)}} \\ \mathcal{E}/J & \xrightarrow[f^{-1}]{} & \mathcal{E}/I \end{array}$$

As in the proof of Theorem 2.7, the vertical functors are monadic. Since the bottom horizontal functor has a left adjoint Σ_f, by the "adjoint lifting theorem‘" (see 4.5.6 in [3]), so does the upper horizontal functor, proving that f^{-1} has a right adjoint. □

The case of sets throws an interesting light on these results. In terms of families of sets, the mapping $f\colon I \longrightarrow J$ yields the pullback functor

$$f^{-1}\colon \mathsf{Set}/J \longrightarrow \mathsf{Set}/I, \quad (B_j)_{j\in J} \mapsto \big(B_{f(i)}\big)_{i\in I}$$

which acts thus by re-indexing the families of sets. In that particular case, the preservation of the exponentiation and the subobject classifier follow then from their pointwise definition, as observed in Proposition 2.8. But the important thing is about the functors Σ_f and π_f.

It is routine to check that the left adjoint Σ_f of f^* is simply

$$\Sigma_f(A_i)_{i\in I} = \left(\coprod_{\{i|f(i)=j\}} A_i\right)_{j\in J}$$

while the right adjoint π_f is given by

$$\pi_f(A_i)_{i\in I} = \left(\prod_{\{i|f(i)=j\}} A_i\right)_{j\in J}.$$

The existence of Σ_f and π_f in the case of an elementary topos is thus some kind of existence of internal coproducts and products.

Corollary 2.10 *In a topos $\mathcal{E}$, consider a morphism $f\colon I \longrightarrow J$ and view it as an object of $\mathcal{E}/J$. The corresponding exponentiation functor $(-)^f$ on $\mathcal{E}/J$ is the composite $\pi_f \circ f^{-1}$.*

Sketch of proof The product in $\mathcal{E}/J$ is obtained by pullback in $\mathcal{E}$:

$$\begin{array}{ccc} B & \longrightarrow & A \\ {\scriptstyle f^{-1}(g)}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ I & \underset{f}{\longrightarrow} & J \end{array}$$

that is,

$$g \times_J f = f \circ f^{-1}(g) = (\Sigma_f \circ f^{-1})(g).$$

By Theorem 2.9, Σ_f and f^{-1} admit respectively f^{-1} and π_f as right adjoints. Thus $\pi_f \circ f^{-1}$ is right adjoint to $- \times_J f$ in $\mathcal{E}/J$. □

A special instance of the action of the functors Σ_f and π_f will play an important role in developing the internal logic of a topos, and in particular the quantifiers. This can be guessed at once from the following proposition, where we reduce our attention to the very special case where f is a projection of a product and we investigate the action of Σ_f and π_f only on monomorphisms.

Proposition 2.11 *In the category of sets, consider $p_B\colon A \times B \longrightarrow B$, the projection of a binary product. Consider further a subset $s\colon S \rightarrowtail A \times B$. Writing Im for the image of a mapping*

$$\begin{aligned} \mathsf{Im}\,\Sigma_{p_B}(s) &= \big(\{b \in B | \exists a \in A \quad (a,b) \in S\} \rightarrowtail B\big) \\ \pi_{p_B}(s) &= \big(\{b \in B | \forall a \in A \quad (a,b) \in S\} \rightarrowtail B\big) \end{aligned}$$

Sketch of proof Since s is injective, $S_{(a,b)} = s^{-1}(a, b)$ is the singleton $\{(a, b)\}$ when $(a, b) \in S$ and is empty otherwise. So all the A_i's in the proof of Theorem 2.9 are singletons or empty sets. In the case of π_{p_B}, the corresponding products are thus singletons if and only if all factors are non-empty. □

In view of proposition 2.11, it is sensible to define:

Definition 2.12 In a topos, consider a projection $p_B \colon A \times B \longrightarrow B$. Consider further a subobject $s \colon S \rightarrowtail A \times B$. We shall write

- $\exists_{p_B}(S)$ for the image of $\Sigma_{p_B}(s)$, as subobject of B;
- $\forall_{p_B}(S)$ for $\pi_{p_B}(s)$, as subobject of B.

2.4 Exactness properties

This section extends to elementary toposes various properties encountered in the case of a Grothendieck topos (see Section 1.6).

Proposition 2.13 *In an elementary topos, all existing colimits are universal.*

Sketch of proof The pullback functors have right adjoints by Theorem 2.9, thus preserve colimits. □

Proposition 2.14 *An elementary topos is a regular and exact category.*

Sketch of proof Let us prove the regularity. Coequalizers exist by Theorem 2.7 and are universal by Proposition 2.13. Thus the topos is a regular category. See Proposition 5.9.6 in [4] for the exactness. □

Corollary 2.15 *In an elementary topos,*

1. *every monomorphism is regular;*
2. *every epimorphism is regular;*
3. *a morphism which is both a monomorphism and an epimorphism is an isomorphism.*

Sketch of proof The subobject classifier $t \colon \mathbf{1} \rightarrowtail \Omega$ admits the trivial retraction $r \colon \Omega \longrightarrow \mathbf{1}$, thus is a regular monomorphism: it is the equalizer of the pair $(\mathsf{id}_\Omega, t \circ r)$. Every monomorphism is then regular, as pullback of t along its characteristic morphism.

By Proposition 2.14, an epimorphism f factors as $f = i \circ p$, with p a regular epimorphism and i a monomorphism. Since f is an epimorphism, so is i. But i is also a regular monomorphism, thus an isomorphism. □

Proposition 2.16 *In an elementary topos*

1. *The initial object* $\mathbf{0}$ *is* strict, *thus is in particular a subobject of every object;*
2. *the pushout of a monomorphism along an arbitrary morphism is still a monomorphism and the pushout square is also a pulback;*

3. *finite coproducts are disjoint;*

4. *finite unions of subobjects exist.*

Sketch of proof The strictness of $\mathbf{0}$ means that every morphism $A \longrightarrow \mathbf{0}$ is an isomorphism. This is the case by universality of the empty colimit (see Proposition 2.13).

We refer to Proposition 5.9.10 in [4] for the second statement. Since a coproduct $A \coprod B$ is the pushout of these two objects over $\mathbf{0}$, the morphisms of the coproduct are thus monomorphisms and the pushout square is also a pullback: this is the so-called *disjointness of coproducts*.

The union of two subobjects $R \rightarrowtail A$ and $S \rightarrowtail A$ is the image of the corresponding factorization $R \coprod S \longrightarrow A$. □

2.5 Heyting algebras in a topos

Let us introduce the notion of a *Heyting algebra*, which is closely related to that of a locale.

Definition 2.17 A *Heyting algebra* is a Cartesian closed lattice with top and bottom element.

As in Theorem 1.5, we shall write $s \Rightarrow -$ to indicate the right adjoint to $- \wedge s$, yielding thus

$$r \wedge s \leq t \ \text{ iff } \ r \leq (s \Rightarrow t).$$

Proposition 2.18 *The locales are exactly the complete Heyting algebras.*

Sketch of proof By Theorem 1.5, every locale is a Heyting algebra. Conversely in a complete Heyting algebra viewed as a category, $- \wedge s$ admits the right adjoint $s \Rightarrow -$, thus preserves all joins. □

Proposition 2.19 *Every Boolean algebra is a Heyting algebra.*

Sketch of proof Simply define $(s \Rightarrow t) = \complement s \vee t$. □

Complements may not exist in a Heyting algebra, but a weaker property holds:

Proposition 2.20 *In a Heyting algebra H, every element u has a* pseudo-complement*, that is, a greatest element $\neg u$ whose meet with u is the bottom element 0.*

Sketch of proof In Definition 2.17, simply put $\neg u = (u \Rightarrow 0)$. □

Of course in a Boolean algebra, the complement of an element is its pseudo-complement.

Example 2.21 In the locale of open subsets of a topological space, $\neg U$ is the interior of the set-complement of U. □

Theorem 2.22 *In an elementary topos, the subobjects of every object constitute a Heyting algebra.*

Sketch of proof Given two subobjects $\sigma\colon S \rightarrowtail A$ and $\tau\colon T \rightarrowtail A$, it remains to prove the existence of the subobject $(S \Rightarrow T) \rightarrowtail A$. Writing φ_R for the characteristic morphism of a subobject $\rho\colon R \rightarrowtail A$, $S \Rightarrow T$ is defined as the following equalizer

$$(S \Rightarrow T) \rightarrowtail A \overset{\varphi_{S\cap T}}{\underset{\varphi_S}{\rightrightarrows}} \Omega.$$

By definition of an equalizer, we have

$$\begin{aligned} R \subseteq (S \Rightarrow T) \quad &\text{iff} \quad \varphi_S \circ \rho = \varphi_{S\cap T} \circ \rho \\ &\text{iff} \quad R \cap S = R \cap (S \cap T) \\ &\text{iff} \quad R \cap S \subseteq T. \end{aligned}$$

which forces the conclusion. □

It is well-know that an algebraic notion like that of a Heyting algebra H can be internalized in every category $\mathcal{C}$ with finite limits:

- giving the top and bottom elements is giving morphisms $\mathbf{1} \longrightarrow H$;
- giving the operations $\wedge$, $\vee$, $\Rightarrow$ is giving morphisms $H \times H \longrightarrow H$.

These data have then to satisfy axioms expressed by the commutativity of some diagrams. Of course $u \leq v$ is translated as the equality $u \wedge v = u$.

Theorem 2.23 *The object Ω of an elementary topos $\mathcal{E}$ is provided with the structure of an internal Heyting algebra. For every object $A \in \mathcal{E}$, this induces by composition a Heyting algebra structure on the set $\mathcal{E}(A, \Omega)$ of morphisms; in terms of corresponding subobjects of A, this is the Heyting algebra structure of Theorem 2.22.*

Sketch of proof The various ingredients for an internal Heyting algebra are defined as follows:

- the top element t (t for true") is the subobject classifier $t\colon \mathbf{1} \rightarrowtail \Omega$, that is, the characteristic morphism of the identity on $\mathbf{1}$;
- the bottom element $f\colon \mathbf{1} \longrightarrow \Omega$, ($f$ for " false") is the characteristic morphism of the zero subobject $\mathbf{0} \rightarrowtail \mathbf{1}$;
- $\wedge\colon \Omega \times \Omega \longrightarrow \Omega$ is the characteristic morphism of the diagonal of Ω, also written $=_\Omega$ in Definition 1.14;
- $\vee\colon \Omega \times \Omega \longrightarrow \Omega$ is the characteristic morphism of the union of the two subobjects $t \times \mathsf{id}_\Omega\colon \mathbf{1} \times \Omega \longrightarrow \Omega \times \Omega$ and $\mathsf{id}_\Omega \times t\colon \Omega \times \mathbf{1} \longrightarrow \Omega \times \Omega$;
- $\Rightarrow\colon \Omega \times \Omega \longrightarrow \Omega$ is the characteristic morphism of the poset structure subobject $\leq_\Omega\ \rightarrowtail \Omega \times \Omega$, that is, of the equalizer of $\wedge$ and p_1 ($a \leq b$ iff $a \wedge b = a$);
- $\neg\colon \Omega \longrightarrow \Omega$, that is $(\bullet \Rightarrow 0)$ (see Corollary 2.20), is then the characteristic morphism of $f\colon \mathbf{1} \longrightarrow \Omega$.

Probably the case of $\vee$ calls a comment. Given subobjects S, T of A with characteristic morphisms φ_S and φ_T, one has the pullbacks

$$
\begin{array}{ccc}
S & \longrightarrow & \mathbf{1}\times\Omega \\
\downarrow & & \downarrow{\scriptstyle t\times \mathsf{id}} \\
A & \xrightarrow[(\varphi_S,\varphi_T)]{} & \Omega\times\Omega
\end{array}
\qquad\qquad
\begin{array}{ccc}
T & \longrightarrow & \Omega\times\mathbf{1} \\
\downarrow & & \downarrow{\scriptstyle \mathsf{id}\times t} \\
A & \xrightarrow[(\varphi_S,\varphi_T)]{} & \Omega\times\Omega
\end{array}
$$

and pulling back preserves unions by Proposition 2.16. The rest is straightforward calculation. □

A last comment. In the localic case, we often thought Ω as an object of *truth values*, when referring to "the biggest level where some property is true ...". Observe further that the internal poset structure of Ω "coincides" somehow with the implication, since

$$(a \Rightarrow b) = 1 \text{ iff } 1 \leq (a \Rightarrow b) \text{ iff } (1 \wedge a) \leq b \text{ iff } a \leq b.$$

2.6 The axiom of infinity

It is possible to develop arithmetic, analysis, differential geometry, and so on, internally in a topos. But of course, not in every topos: you cannot possibly imagine developing calculus ... in the topos of finite sets! For doing so, you badly need to use "infinite objects", whose existence – just like in set-theory – must be attested by a corresponding "axiom of infinity".

Among various possible axioms of infinity, all equivalent in a topos, we choose the elegant categorical characterization of the set of natural numbers in Set, since this makes pertinent sense in more general categories than just toposes.

Definition 2.24 By a *Natural Number Object* in a category with a terminal object, is meant a triple $(\mathbb{N}, 0, s)$ as in the following diagram

$$
\begin{array}{ccccc}
\mathbf{1} & \xrightarrow{0} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\
 & {\scriptstyle u}\searrow & \vdots{\scriptstyle w} & & \vdots{\scriptstyle w} \\
 & & A & \xrightarrow[v]{} & A
\end{array}
$$

with the property that given another such triple (A, u, v), there exists a unique morphism w making the diagram commutative.
An elementary topos satisfies the *axiom of infinity* when it contains a Natural Number Object.

Clearly, by its universal property, a Natural Number Object is necessarily unique up to an isomorphism.

Examples 2.25 Examples of Natural Number Objects in toposes.

1. In the topos of sets, choose $\mathbb{N}$ to be the set of natural numbers, together with the choice of the number 0 and $s(n) = n + 1$, the "successor" mapping. Then w is defined inductively by $w(0) = u$ and $w(n + 1) = v\big(w(n)\big)$.

2. In a topos of presheaves, the Natural Number Object is defined pointwise as in Set.

3. In a Grothendieck topos $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$, apply the associated sheaf functor to the Natural Number Object in $\mathrm{Pr}(\mathcal{C})$. □

Suggestion(s) for further reading

Once you have a Natural Number Object, you can start developing arithmetic. For example, the addition $+\colon \mathbb{N}\times\mathbb{N}\longrightarrow\mathbb{N}$ corresponds by Cartesian closedness to the morphism α in the following diagram

$$\begin{array}{ccccc} \mathbf{1} & \xrightarrow{\;0\;} & \mathbb{N} & \xrightarrow{\;s\;} & \mathbb{N} \\ & \searrow^{i} & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ & & \mathbb{N}^{\mathbb{N}} & \xrightarrow[s^{\mathbb{N}}]{} & \mathbb{N}^{\mathbb{N}} \end{array}$$

where i corresponds by Cartesian closedness to the identity on $\mathbb{N}$. One can, in an analogous way, define the multiplication on $\mathbb{N}$, but also the ordering on $\mathbb{N}$. It turns out that a Natural Number Object in a topos satisfies, in the internal logic of the topos, all the axioms of Peano's arithmetic (see Theorem 8.1.13 in [4]).

In a topos with Natural Number Object, it is routine to define the object $\mathbb{Q}$ of rational numbers. One can also define further an object $\mathbb{R}$ of real numbers, using Cauchy sequences or Dedekind cuts ... but these two constructions do no longer, in general, produce isomorphic results.

One can also define infinite objects in an arbitrary elementary topos (see Definition 8.2.2 in [4]): as an example, an object A which is isomorphic to $A \amalg \mathbf{1}$ is infinite. In a topos, the existence of an infinite object is equivalent to the existence of a Natural Number Object (see Proposition 8.2.5 in [4]). But this is a very peculiar property of toposes: for example in an abelian category, since $\mathbf{1}$ is the zero object, $A \amalg \mathbf{1} \cong A \oplus \mathbf{0} \cong A$ for every object A of the abelian category.

The definition of finite objects is more subtle. Once more in a topos, there are various non-equivalent approaches to the notion of finiteness (see Section 8.5 in [4]), some of them requiring the existence of a Natural Number Object, like finite cardinality, some of them making sense in every topos, like Kuratowski finiteness, which (intuitively) expresses the possibility of reconstructing the whole object by binary unions, starting from the empty subobject and the singletons.

Lesson 3

Internal topologies and sheaves

When passing from the topos of presheaves on a locale L, to the topos of sheaves on L, one inherits an additional property: the possibility of gluing compatible families. Analogously, when passing from the topos of presheaves on a small category $\mathcal{C}$ to the topos of sheaves on a site $(\mathcal{C}, \mathcal{T})$, additional extension properties become valid. The same kind of technique can be developed in an elementary topos, in order to exhibit a subtopos with additional properties. As an example, we shall show how to construct, from an arbitrary topos, a subtopos where the lattices of subobjects are Boolean algebras, like in Set, and not just Heyting algebras.

3.1 Internal topologies

Our purpose is now to internalize the notions of Grothendieck topology and sheaf in an elementary topos.

Let $(\mathcal{C}, \mathcal{T})$ be a site. Proposition 1.23 tells us in particular that the subobject classifier of the corresponding topos $\mathsf{Pr}(\mathcal{C})$ of presheaves is given by

$$\Omega(C) = \{S | S \text{ is a sub-presheaf of } \mathcal{C}(-, C)\}.$$

The action of the presheaf Ω on the arrows of $\mathcal{C}$ is just pulling back: given $f\colon D \longrightarrow C$

$$\Omega(f)(S) = \mathcal{C}(-, f)^{-1}(S).$$

By condition 3 in Definition 1.14, when S covers C, then $\Omega(f)(S) = \mathcal{C}(-, f)^{-1}(S)$ covers D. This is precisely saying that $\mathcal{T}$ is a subpresheaf of Ω. With a Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$ corresponds thus a morphism $j\colon \Omega \longrightarrow \Omega$ in the topos of presheaves, namely, the characteristic morphism of $\mathcal{T}$, viewed as a subobject of Ω.

Theorem 3.1 *Let $\mathcal{C}$ be a small category. There exists a bijection between*

1. *the Grothendieck topologies $\mathcal{T}$ on $\mathcal{C}$;*
2. *the morphisms $j\colon \Omega \longrightarrow \Omega$, in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves, which make commutative the following three diagrams:*

$$\begin{array}{ccc} \mathbf{1} & & \\ {\scriptstyle t}\downarrow & \searrow{\scriptstyle t} & \\ \Omega & \xrightarrow{\;j\;} & \Omega \end{array} \qquad \begin{array}{ccc} \Omega & \xrightarrow{\;j\;} & \Omega \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle j} \\ & & \Omega \end{array} \qquad \begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\;j \times j\;} & \Omega \times \Omega \\ {\scriptstyle \wedge}\downarrow & & \downarrow{\scriptstyle \wedge} \\ \Omega & \xrightarrow{\;j\;} & \Omega \end{array}$$

Sketch of proof We have just seen how to construct j from $\mathcal{T}$.

To see that j, constructed from $\mathcal{T}$, makes commutative the three diagrams of the statement, let us recall the form of that characteristic morphism j (see the proof of Proosition 1.25)

$$j_C(S)(D) = \{g\colon D \to C \big| \mathcal{C}(-,g)^{-1}(S) \in \mathcal{T}(D)\}.$$

Given a Grothendieck topology $\mathcal{T}$, the first diagram in the statement rephrases the fact that $\mathcal{C}(-,C)$ covers C. The second diagram holds by the third axiom in 1.14. The third diagram, which does not have its direct counterpart in Definition 1.14, commutes, essentially because the intersection of two covering sieves is covering, as follows easily from Definition 1.14.

Conversely, consider a morphism j making commutative the three diagrams indicated and let $\mathcal{T}$ be the subobject of Ω classified by j. The first axiom in Definition 1.14 is just the commutativity of the first diagram, while the second axiom rephrases the fact that $\mathcal{T}$ is a subobject of Ω. For the third axiom in Definition 1.14, and with its notation,

$$j_C(S)(D) = \{g\colon D \to C \big| \mathcal{C}(-,g)^{-1}(S) \in \mathcal{T}(D)\} \supseteq R(D)$$

by assumption on S and R. Thus $R \subseteq j_C(S)$ and therefore

$$j_C(R) = j_C\big(R \cap j_C(S)\big) = j_C(R) \cap j_C j_C(S) = j_C(R) \cap j_C(S)$$

proving that $j_C(R) \subseteq j_C(S)$. But $j_C(R) = \mathcal{C}(-,C)$, thus the same holds for S. $\square$

With in view Theorem 3.1, we define then:

Definition 3.2 A Lawvere-Tierney *topology* in a topos $\mathcal{E}$ is a morphism $j\colon \Omega \longrightarrow \Omega$ satisfying

$$j \circ t = t, \quad j \circ j = j, \quad j \circ \wedge = \wedge \circ (j \times j)$$

(see the diagrams in Theorem 3.1).

3.2 Internal sheaves

A topology on Ω induces a corresponding closure operator on subobjects:

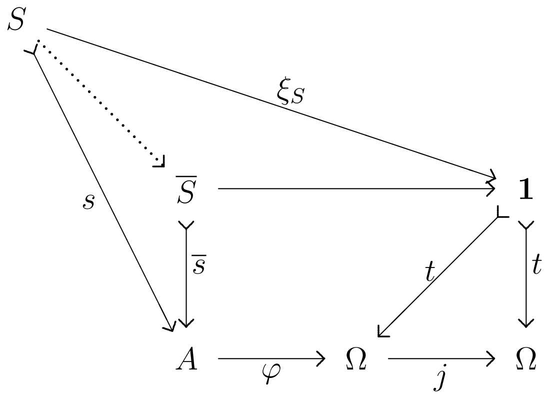
Definition 3.3 Let $\mathcal{E}$ be a topos and $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$. Given a subobject $S \rightarrowtail A$ with characteristic morphism $\varphi\colon A \longrightarrow \Omega$, the subobject $\overline{S} \rightarrowtail A$ classified by $j \circ \varphi$ is called the *j-closure* of S.

Let us now recall the axioms for a universal closure operator (see Section 5.7 in [2]). For all subobjects R, S of A and every morphism $f\colon B \longrightarrow A$:

1. $S \subseteq \overline{S}$;
2. $S \subseteq T \implies \overline{S} \subseteq \overline{T}$;
3. $\overline{\overline{S}} = S$;
4. $f^{-1}\big(\overline{S}\big) = \overline{f^{-1}(S)}$.

Proposition 3.4 *Every topology on the object Ω of a topos induces a universal closure operator.*

Sketch of proof This is just routine calculation. See Section 9.1 in [4]. For example, considering the following diagram, where φ is the characteristic morphism of S,



$$j\varphi s = jt\xi_S = t\xi_S$$

from which the expected factorization $S \rightarrowtail \overline{S}$ through the pullback. $\square$

Definition 3.5 Let $\mathcal{E}$ be a topos and $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$. Given a subobject $S \rightarrowtail A$

1. S is *closed* in A when $\overline{S} = S$;
2. S is *dense* in A when $\overline{S} = A$.

Proposition 3.6 *Let j be a topology in a topos. Dense subobjects and closed subobjects are stable under pullbacks.*

Sketch of proof By Proposition 3.4.4. $\square$

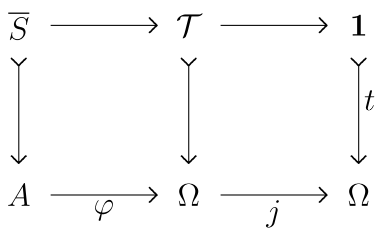
Proposition 3.7 *Let j be a topology in a topos. Then j is the characteristic morphism of the closure of the subobject classifier $t\colon \mathbf{1} \rightarrowtail \Omega$.*

Sketch of proof The characteristic morphim of $t\colon \mathbf{1} \rightarrowtail \Omega$ is the identity on Ω, thus $j = j \circ \mathsf{id}_\Omega$ is the characteristic morphism of its closure. $\square$

Let us now describe the closure operation in the case of a Grothendieck topos.

Example 3.8 *Consider a site $(\mathcal{C}, \mathcal{T})$ and the corresponding topology $j\colon \Omega \longrightarrow \Omega$ in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves (see Theorem 3.1) . Given a sub-presheaf $S \subseteq A$, an element $a \in A(C)$ lies in $\overline{S}(C)$ when $A(f)(a) \in S(D)$, for all the morphisms $f\colon D \longrightarrow C$ of a covering sieve $R \in \mathcal{T}(C)$.*

Sketch of proof $\overline{S}$ is given by the following pullbacks, where φ is the characteristic morphism of $S \rightarrowtail A$:



We have (see the proof of Proposition 1.25)

$$\varphi_C(a)(D) = \big\{ f\colon D \to C \big| A(f)(a) \in S(D) \big\}),$$

By the left hand pullback square, $a \in \overline{S}(C)$ when this sieve is in $\mathcal{T}(C)$. $\square$

Lemma 3.9 *Consider a site $(\mathcal{C}, \mathcal{T})$ and, applying Theorem 3.1, the corresponding topology $j\colon \Omega \longrightarrow \Omega$ in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves. The dense subobjects of a representable presheaf are exactly its covering sieves.*

Sketch of proof Consider a sieve R on C and its characteristic morphism φ.

$$
\begin{array}{ccc}
R & \longrightarrow & \mathbf{1} \\
{\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle t} \\
\mathcal{C}(-,C) & \underset{\varphi}{\longrightarrow} & \Omega
\end{array}
\qquad\qquad
\begin{array}{ccc}
\overline{R} & \longrightarrow & \mathcal{T} \\
{\scriptstyle \overline{r}}\big\downarrow & & \big\downarrow \\
\mathcal{C}(-,C) & \underset{\varphi}{\longrightarrow} & \Omega
\end{array}
$$

The morphism φ corresponds, by the Yoneda lemma, to $R \in \Omega(C)$. In the right hand pullback, φ factors through $\mathcal{T}$, that is $R \in \mathcal{T}(C)$, if and only if $\overline{r}$ is an isomorphism, that is, R is dense in $\mathcal{C}(-,C)$. □

Lemma 3.10 *Consider a site $(\mathcal{C}, \mathcal{T})$ and, applying Theorem 3.1, the corresponding topology $j\colon \Omega \longrightarrow \Omega$ in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves. Every sheaf on $(\mathcal{C}, \mathcal{T})$ is orthogonal to every dense subpresheaf.*

Sketch of proof Given a j-dense subobject $s\colon S \rightarrowtail F$, write F as a colimit of representable functors. By universality of colimits in $\mathsf{Pr}(\mathcal{C})$ and pullback stability of dense subobjects, one can as well write s as a colimit of dense sieves. The rest is routine. □

Definition 1.14 and Lemmas 3.9 and 3.10 suggest to define further:

Definition 3.11 Let $\mathcal{E}$ be a topos and $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$. An object $F \in \mathcal{E}$ is called a j-sheaf when it is orthogonal to every j-dense subobject.

Theorem 3.12 *Let $\mathcal{E}$ be a topos and $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$.*

1. *The full subcategory Sh_j of j-sheaves is a topos.*
2. *The Ω-object of Sh_j is the image Ω_j of j in $\mathcal{E}$.*
3. *Exponentiation in Sh_j is computed as in $\mathcal{E}$.*
4. *The inclusion $\mathsf{Sh}_j \hookrightarrow \mathcal{E}$ has a left adjoint preserving finite limits. This adjoint is called the* associated sheaf functor

Sketch of proof Assertions 2 and 3 describe explicitly the topos structure of the category of sheaves. Let me also explain the construction the sheaf associated with a presheaf A. We consider first the image factorization of j in $\mathcal{E}$

$$\Omega \xrightarrow{\;p\;}\!\!\!\!\rightarrow \Omega_j \rightarrowtail\!\xrightarrow{\;i\;} \Omega.$$

We construct next the commutative parallelogram below

$$
\begin{array}{ccccc}
A & \xrightarrow{\;q\;}\!\!\!\!\rightarrow & \alpha(A) & \rightarrowtail & \overline{\alpha(A)} \\
& {\scriptstyle \{\cdot\}_A}\searrow & & \searrow{\scriptstyle k} & \big\downarrow \\
& & \Omega^A & \xrightarrow[p^A]{} & \Omega_j^A
\end{array}
$$

where thus $k \circ q$ is the image factorization of $p^A \circ \{\cdot\}_A$. The sheaf $a(A)$ associated with A is the closure of $\alpha(A)$ in Ω_j^A. See Sections 9.2 and 9.3 of [4] for the details of the proof. □

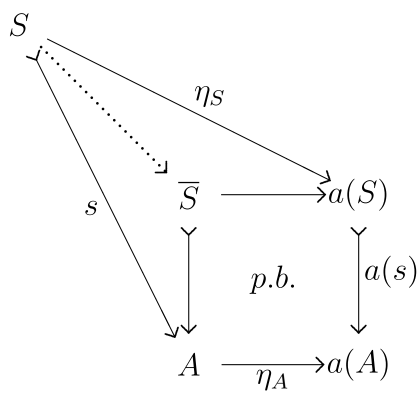
One can further improve that theorem:

Proposition 3.13 *Let $\mathcal{E}$ be an elementary topos. There is a bijection between:*

1. *the topologies on Ω;*
2. *the universal closure operators on $\mathcal{E}$;*
3. *the localizations of $\mathcal{E}$.*

Sketch of proof See [4], Proposition 9.3.9. Here, localization means a full reflective subcategory whose reflection is left exact.

Given a topology j, Proposition 3.4 gives the corresponding closure operator. Given a closure operator, define j as the characteristic morphism of the closure of $t\colon \mathbf{1} \rightarrowtail \Omega$ (see Proposition 3.7). By Theorem 3.12.4, the sheaves for that topology constitute a localization. And given a localization $a \dashv i$ of the topos, define the closure of a subobject $s\colon S \rightarrowtail A$ by the pullback in the following diagram



where η is the unit of the adjunction. □

3.3 Boolean toposes

Let us first observe that

Proposition 3.14 *A Heyting algebra H is a Boolean algebra when $a \vee \neg a = 1$ for every element $a \in H$.*

Sketch of proof Since we have already $a \wedge \neg a = 0$. □

Definition 3.15 A topos $\mathcal{E}$ is *Boolean* when the internal Heyting algebra Ω is an internal Boolean algebra.

Booleanity is a very strong requirement on a topos, an axiom which is almost always avoided. In the case of sheaves on a topological space, this means thus that the lattice of open subsets is Boolean, that is, every open subset is closed.

Proposition 3.16 *In a Boolean topos $\mathcal{E}$, the lattice of subobjects of every object is a Boolean algebra.*

Sketch of proof By Theorem 2.23. □

Theorem 3.17 *A topos is Boolean if and only if* $\Omega \cong \mathbf{1} \amalg \mathbf{1}$.

Sketch of proof In the pullback

$$\begin{array}{ccc} \mathbf{0} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow{\scriptstyle t} \\ \mathbf{1} & \xrightarrow[f]{} & \Omega \end{array}$$

the "false" morphism f is a monomorphism, because $\mathbf{1}$ is terminal. And since the pullback of t and f is $\mathbf{0}$, the union of these two subobjects is a disjoint union, that is, a coproduct. Thus $(t, f): \mathbf{1} \amalg \mathbf{1} \rightarrowtail \Omega$ is a monomorphism, the union of the two subobjects t and f. But f is the pseudo-complement of t. Therefore when Ω is a Boolean algebra, f becomes the complement of t. In that case, the union of the two subobjects t and f is Ω and thus (t, f) is an isomorphism.

The converse implication is quite trivial: $\mathbf{1}\amalg\mathbf{1}$ is a Boolean algebra in every topos. Writing t and f for the two injections of the coproduct, defining the meet or join operation is defining a morphism

$$(\mathbf{1} \amalg \mathbf{1}) \times (\mathbf{1} \amalg \mathbf{1}) \longrightarrow (\mathbf{1} \amalg \mathbf{1}).$$

which, for the sake of precision, we better write as

$$(\mathbf{1}_t \amalg \mathbf{1}_f) \times (\mathbf{1}_t \amalg \mathbf{1}_f) \longrightarrow (\mathbf{1}_t \amalg \mathbf{1}_f).$$

By Cartesian closedness, the product commutes with coproducts, so that we have to define a morphism

$$(\mathbf{1}_t \times \mathbf{1}_t) \amalg (\mathbf{1}_t \times \mathbf{1}_f) \amalg (\mathbf{1}_f \times \mathbf{1}_t) \amalg (\mathbf{1}_f \times \mathbf{1}_f) \longrightarrow (\mathbf{1}_t \amalg \mathbf{1}_f)$$

The $\wedge$ and $\vee$ operations are then defined componentwise via the obvious rules

$$t \wedge t = t, \quad t \wedge f = f, \quad f \wedge t = f, \quad f \wedge f = f$$

$$t \vee t = t, \quad t \vee f = t, \quad f \vee t = t, \quad f \vee f = f.$$

The rest is routine. See Proposition 7.2.2 in [4]. □

Booleanity of the topos is thus the internal version of the so-called *law of excluded middle*: the object Ω of truth values is the disjoint union of true and false.

Examples 3.18 The following are examples of Boolean toposes:

1. the topos of sets;
2. the topos of sheaves on a complete Boolean algebra;
3. given a group G, the topos of G-sets;

Sketch of proof In the second example, each $\Omega(u)$ is a Boolean algebra, where the complement of $v \in \downarrow u$ is $\complement v \wedge u$.

In the third case, Corollary 1.26 shows that $\Omega = \{\emptyset, G\}$ is the two-point Heyting algebra, which is thus trivially a Boolean algebra.

The last example generalizes the third one: see [4], Example 7.2.4. □

Proposition 3.19 *If $\mathcal{E}$ is a Boolean topos all the slice toposes $\mathcal{E}/I$ are Boolean as well.*

Sketch of proof Just by the form of the Ω-object in $\mathcal{E}/I$, described in the proof of Proposition 2.8. $\square$

Suggestion(s) for further reading

Generalizing Example 3.18.3, the topos of presheaves on a small category $\mathcal{C}$ is Boolean if and only if $\mathcal{C}$ is a groupoid (see Example 7.2.4 in [4]).

3.4 Double-negation sheaves

Let us now prove that every topos contains a Boolean part.

Theorem 3.20 *In a topos, the* double negation *morphism* $\neg\neg\colon \Omega \longrightarrow \Omega$ *is a topology and the corresponding subtopos of sheaves is Boolean.*

Sketch of proof Let us first observe that the double negation is a topology (in the sense of Definition 3.2) on every Heyting algebra H in the topos of sets. And let me so take a chance to motivate you not to give up in front of the technical aspects of my next lesson.

First, $\neg 1 = 0$ and $\neg 0 = 1$, thus $\neg\neg 1 = 1$.

Second, $a \leq \neg b$ if and only if $a \wedge b = 0$. Putting $a = \neg c$, if $b \leq c$, we have $\neg c \wedge b \leq \neg c \wedge c = 0$, thus $\neg c \leq \neg b$. So $\neg$ reverses the ordering and therefore, $\neg\neg$ preserves the ordering. Putting $b = \neg a$ in $a \wedge b = 0$, we get $a \leq \neg\neg a$; putting further $a = \neg x$, we obtain $\neg x \leq \neg\neg\neg x$. But since $\neg$ reverses the ordering, from $x \leq \neg\neg x$ we get $\neg\neg\neg x \leq \neg x$. So finally, $\neg\neg\neg = \neg$ and $(\neg\neg) \circ (\neg\neg) = \neg\neg$.

Third, since $\neg\neg$ preserves the ordering, $\neg\neg(x \wedge y) \leq \neg\neg x \wedge \neg\neg y$. Conversely

$$\neg(x \wedge y) \wedge x \wedge y = 0 \;\Rightarrow\; \neg(x \wedge y) \wedge x \leq \neg y = \neg\neg\neg y \;\Rightarrow\; \neg(x \wedge y) \wedge x \wedge \neg\neg y = 0.$$

The same trick can be applied to x, yielding

$$\neg(x \wedge y) \wedge x \wedge \neg\neg y = 0 \;\Rightarrow\; \neg(x \wedge y) \wedge \neg\neg y \leq \neg x = \neg\neg\neg x \;\Rightarrow\; \neg(x \wedge y) \wedge \neg\neg y \wedge \neg\neg x = 0$$

that is finally, $\neg\neg y \wedge \neg\neg x \leq \neg\neg(x \wedge y)$, thus the equality.

Thus $\neg\neg$ is a topology on H. I know: this is just a proof in the case of a set theoretical Heyting algebra H in the topos of sets: not a proof on the Ω-object of an elementary topos. But yes it is! Our next lesson on the internal logic of a topos will tell us that the argument above is an actual proof for every Heyting algebra in every elementary topos, thus in particular on Ω (see Theorem 2.23). I hope that this example will motivate you to get interested in the internal logic of a topos.

We have now to prove that the topos of $\neg\neg$-sheaves is Boolean. That is, in view of Theorem 3.12.2, we must prove that the image of $\neg\neg$ is a Boolean algebra. Once more, I do it for a set-theoretical Heyting algebra H in the topos of sets. We consider thus

$$H_{\neg\neg} = \{\neg\neg x | x \in H\} = \{x \in H | x = \neg\neg x\}$$

since $\neg\neg\neg\neg = \neg\neg$.

By the third axiom for a topology, the meet of two elements of $H_{\neg\neg}$ is their meet in H. On the other hand it is immediate that the join of two elements in $H_{\neg\neg}$ is, in terms of the operations of H, $\neg\neg(x \vee y)$.

Since $\neg\neg\neg = \neg$, $x \in H_{\neg\neg}$ implies $\neg x \in H_{\neg\neg}$. Let us prove that $\neg x$ is the complement of x in $H_{\neg\neg}$. We have of course $x \wedge \neg x = 0$. It remains to prove that $\neg\neg(x \vee \neg x) = 1$. For that, it suffices to prove that $y = \neg(x \vee \neg x) = 0$. Since $y \wedge (x \vee \neg x) = 0$, by distributivity we have $(y \wedge x) \vee (y \wedge \neg x) = 0$, thus both $y \wedge x = 0$ and $y \wedge \neg x = 0$. This second equality implies $y \leq \neg\neg x = x$ and putting that in the first equality, we obtain $y = 0$.

Yes, yes: by the internal logic of a topos, that's again a proof that for every Heyting algebra H in every elementary topos, the image of $\neg\neg$ is a Boolean algebra. So this is in particular the case for Ω. □

3.5 The axiom of choice

The axiom of choice in a topos is worth a comment ... even if this is the kind of axiom that you definitely do not want to assume in a topos.

The axiom of choice says that given a family $(A_i)_{i \in I}$ of non-empty sets, it is possible to construct a set by picking up one element a_i in each A_i. Put $A = \coprod_{i \in I} A_i$ and write $p\colon A \longrightarrow I$ for the projection sending all the elements of A_i on i. Saying that the A_i's are non-empty is saying that p is surjective. Picking up an element a_i in each A_i is choosing a section s of p and taking its image. Therefore we define:

Definition 3.21 A topos $\mathcal{E}$ satisfies the *axiom of choice* when every epimorphism admits a section.

Suggestion(s) for further reading

The following properties hold concerning the axiom of choice (see Section 7.5 in [4]):

1. A topos satisfying the axiom of choice is Boolean.

2. A Grothendieck topos satisfying the axiom of choice is localic.

3. A localic topos satisfies the axiom of choice if and only if it is Boolean.

Of course in the last assertion, the axiom of choice must be assumed in Set.

Booleanity does not imply the axiom of choice, not even for Grothendieck toposes, as the well-known case of sets already indicates. And even assuming the axiom of choice for sets, given a non-trivial group G, the topos of G-sets is Boolean but does not satisfy the axiom of choice.

Lesson 4

Internal logic of a topos

In this lesson "topos" always means "elementary topos", except otherwise specified.

4.1 The language of a topos

Suppose you want to study the field of real numbers. You will have to handle "actual numbers" like 5, $\frac{2}{3}$, π, $\sqrt{2}$, and so on. We call these *constants of type* $\mathbb{R}$: there are thus as many such constants as real numbers. But you will also have to handle formulæ like

$$a \times (b + c) = (a \times b) + (a \times c)$$

where a, b, c stand now for arbitrary, unspecified real numbers. We call a, b, c *variables of type* $\mathbb{R}$. Since a formula which you can write is a finite sequence of symbols, you only need each time a finite (possibly very large) number of such variables ... thus it suffices to give yourself a denumerable set of variables of type $\mathbb{R}$ in order to be able to write down all possible formulæ about real numbers. The number of variables has thus nothing to do with the number of elements of $\mathbb{R}$. And notice that even if you want to speak of the singleton ... you can possibly want to use two distinct variables in order to express in the following way the fact that the singleton has only one element

$$\forall x \; \forall y \; x = y.$$

The language of a topos is the collection of all "well formed" successions of symbols that you are allowed to use: the point is just, for the time being, to know what "well formed" means, not to give an actual meaning to what you write. For example in the English language, you are allowed to use all the words which appear as entries in a dictionary and combine them using the rules appearing in an English grammar. So that

The prolific fork mixes yellow mathematics.

is a perfectly well-formed English sentence.

Definition 4.1 *The* language of a topos $\mathcal{E}$ *consists in giving, for every object* $A \in \mathcal{E}$

- *a formal symbol, called a* constant of type A*, for every arrow* $\mathbf{1} \longrightarrow A$*;*
- *a denumerable set of formal symbols, called the* variables of type A*;*

together with the various formal expressions, called terms *and* formulæ*, constructed from these, as in Definitions 4.2 and 4.3 below.*

From now on, we shall intuitively think the constants and the variables of type A – and more generally, all terms of type A as in Definition 4.2 – as "formal internal elements" of A. Formal, because up to now, they are just living in our imagination; they do not (yet) correspond to any object or arrow in the topos. Intuitively, we want to think the objects of the topos as kinds of generalized sets. For example in a localic topos, a sheaf is a kind of generalized set with elements "at various levels". In this spirit, if A is provided with, for example, an addition $+\colon A \times A \longrightarrow A$, we certainly want to be able to speak of the formal internal element $a + b$, with a, b variables or constants of type A. And if $f\colon A \longrightarrow B$ is a morphism, we want to be able to speak of the formal internal element $f(a)$ of type B. And so on. These additional formal elements of type A or B that we want to handle, will be called the *terms* of type A or B. Definition 4.2 will describe inductively all the possible terms that we can consider in a topos: thus all the possible formal expressions that we can think of, as representing intuitively "formal internal elements".

Next, suppose that A is in fact provided with the structure of an internal group. The addition is thus associative and we want of course to be able to express this associativity by saying that $(a+b)+c = a+(b+c)$ for all formal internal elements a, b, c of A. Such a formal expression between internal elements is no longer a formal element: it is a formula which intuitively expresses a formal property involving terms like $a + b$, $b + c$ and so on ... but a formula which, up to now, does not (yet) have any meaning in the topos. Definition 4.3 will describe inductively all the formulæ that we want to possibly use in a topos.

An additional comment. In the formulæ, we shall want to use mathematical symbols like $\wedge$, $\vee$, $\implies$, $\exists a$ of $\forall a$, and so on. As in classical logic, a variable a which appears in a quantifier $\exists a$ or $\forall a$, is called a *bound* variable. A variable which does not appear in a quantifier is called a *free* variable.

And before going to the definition of terms and formulæ, it is probably useful to recall once more that in a topos, the "constants" of type Ω^A, that is, the morphisms $\mathbf{1} \longrightarrow \Omega^A$, are in bijection with the morphisms $A \longrightarrow \Omega$ and thus further with the subobjects of A. Thus Ω^A should be thought formally as the "object of subobjects of A": a constant of type Ω^A represents an actual subobjects of A; a variable of type Ω^A should be thought as representing a formal internal subobject of A.

To avoid hiding the spirit of the following definitions behind unessential details concerning the sets of free variables, we omit these details and refer the reader to Definition 6.1.1 in [4] for more precision.

Definition 4.2 *In a topos $\mathcal{E}$, the terms of the internal language are the formal expressions defined inductively by:*

1. *the constants of type A are terms of type A;*
2. *the variables of type A are terms of type A;*
3. *if τ is a term of type A and $f\colon A \longrightarrow B$ is a morphism in $\mathcal{E}$, $f(\tau)$ is a term of type B;*
4. *if $\tau_1, \dots, \tau_n$ are terms of respective types $A_1, \dots, A_n$, then $(\tau_1, \dots, \tau_n)$ is a term of type $A_1 \times \cdots \times A_n$;*
5. *if τ is a term of type A with free variables $a_1, \dots, a_n$ of respective types $A_1, \dots, A_n$; if $\sigma_1, \dots, \sigma_n$ are terms of respective types $A_1, \dots, A_n$, not containing any bound variable of τ, then $\tau(\sigma_1, \dots, \sigma_n)$ remains a term of type A;*

6. *if φ is a formula with free variables $a_1, \ldots, a_n$, $b_1, \ldots, b_m$, of respective types $A_1, \ldots, A_n$, $B_1, \ldots, B_m$*
$$\big\{(a_1, \ldots, a_n)\big|\varphi(a_1, \ldots, a_n, b_1, \ldots, b_m)\big\}$$
is a term of type $\Omega^{A_1 \times \ldots \times A_n}$.

Definition 4.3 *In a topos $\mathcal{E}$, the formulæ of the internal language are the formal expressions defined inductively by:.*

1. *the symbols* true *and* false *are formulæ;*

2. *if τ and σ are terms of type A, then $\tau = \sigma$ is a formula.*

3. *if τ is a term of type A and Σ is a term of type Ω^A, then $\tau \in \Sigma$ is a formula;*

4. *if φ is a formula, then $\neg\varphi$ is a formula;*

5. *if φ and ψ are formulæ, then $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \Rightarrow \psi$ are formulæ;*

6. *if φ is a formula with free variables $a, b_1, \ldots, b_n$ of respective types $A, B_1, \ldots, B_n$, then*
$$\exists a \ \varphi(a, b_1, \ldots, b_n), \quad \forall a \ \varphi(a, b_1, \ldots, b_n)$$
are formulæ with free variables $b_1, \ldots, b_n$;

7. *if φ is a formula with free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$ and $\sigma_1, \ldots, \sigma_n$ are terms of respective types $A_1, \ldots, A_n$ with the same free variables $b_1, \ldots, b_m$ of respective types $B_1, \ldots, B_m$, then*
$$\varphi\big(\sigma_1(b_1, \ldots, b_m), \ldots, \sigma_n(b_1, \ldots, b_m)\big)$$
is a formula with free variables $b_1, \ldots, b_m$.

Notice that assuming that two formulæ or terms have the same free variables is not really a restriction, because it cannot hurt to consider a given variable as a free variable of a formula or a term ... even if it does not explicitly appear in the formula or the term.

Of course as usual, $\exists! x\, \varphi(x)$ is an abbreviation for
$$\big(\exists x\, \varphi(x)\big) \wedge \big((\varphi(y) \wedge \varphi(z)) \Rightarrow (y = z)\big).$$

In other words, this section can be summarized by saying that the language of a topos mimics exactly the usual language of set theory. But up to now, it is just a kind of "scrabble" game where you construct "sentences" (formulæ or terms) by juxtaposing formal symbols; but not all "sentences" are acceptable: only those appearing in the "dictionaries" and "grammars" of Definitions 4.2 and 4.3. And thus the "sentence"
$$\exists a \ \forall b \ \big(\neg(a = a)\big) \wedge (a = b)$$
is a perfectly (stupid) well-formed sentence of our language ...

4.2 Realization of terms and formulæ

Let us go back to our example of real numbers. Given a rational number a and a natural number b, we can construct the real number $b^{a\pi}$. In the language of Section 4.1, $b^{a\pi}$ is thus a term of type $\mathbb{R}$ with two variables a, b of respective types $\mathbb{Q}$ and $\mathbb{N}$. Such a term induces thus a mapping

$$\mathbb{Q} \times \mathbb{N} \longrightarrow \mathbb{R}, \quad (a, b) \mapsto b^{a\pi}.$$

That mapping is what we shall call the *realization of the term* $b^{a\pi}$. Doing so, we associate, with the formal term $b^{a\pi}$ of type $\mathbb{R}$ in Set, a well-defined morphism in the topos of sets.

In exactly the same spirit, we shall associate, with every formal term in a topos, an actual morphism of the topos. With every term τ of type A with free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$, we shall associate a "realization" morphism

$$\ulcorner \tau \urcorner \colon A_1 \times \cdots \times A_n \longrightarrow A.$$

Again the definition proceeds inductively. For short, we use without recalling them the notation and numeration of Definition 4.2.

Definition 4.4 *In a topos $\mathcal{E}$, with the notation and the numeration of Definition 4.2, the* realization $\ulcorner \tau \urcorner$ *of a term τ is defined inductively by:*

1. *the realization of a constant is the constant itself;*
2. *the realization of a variable of type A is the identity on A;*
3. $\ulcorner f(\tau) \urcorner = f \circ \ulcorner \tau \urcorner$*;*
4. $\ulcorner (\tau_1, \ldots, \tau_n) \urcorner = (\ulcorner \tau_1 \urcorner, \ldots, \ulcorner \tau_n \urcorner)$
5. $\ulcorner \tau(\sigma_1, \ldots, \sigma_n) \urcorner = \ulcorner \tau \urcorner \circ (\ulcorner \sigma_1 \urcorner, \ldots, \ulcorner \sigma_n \urcorner)$*;*
6. $\ulcorner \{(a_1, \ldots, a_n) | \varphi\} \urcorner$ *is the morphism $B_1 \times \cdots \times B_m \longrightarrow \Omega^{A_1 \times \cdots \times A_n}$ corresponding to $\ulcorner \varphi \urcorner$ by Cartesian closedness (see Definition 4.5).*

Next, going back once more to real numbers in Set, consider, for three variables of type $\mathbb{R}$, the formula $a + b = c$. This yields at once a mapping

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \{\mathsf{false}, \mathsf{true}\}$$

which maps each concrete triple (a, b, c) of real numbers on the "truth value" of $a + b = c$. This mapping is what we shall call the *truth table* of the formula $a + b = c$. Let us recall that in the topos of sets, $\{\mathsf{false}, \mathsf{true}\}$ is precisely the object Ω, the subobject classifier. And the subobject classified by this *truth table* is precisely the set of those triples (a, b, c) such that $a + b = c$.

We have already observed several times that in a localic topos, the object Ω somehow plays the role of "object of truth values". For example considering the equality morphism $= \colon A \times A \longrightarrow \Omega$ of Definition 2.3, given a, b in $A(u)$, the element $=_u (a, b)$ is the "truth value of $a = b$", the highest level where the restrictions of a and b become equal. The morphism $=$ is thus a kind of " truth table" of the formula $a = b$. In this spirit we shall now associate, with every formula φ with free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$, a "truth table" morphism

$$\ulcorner \varphi \urcorner \colon A_1 \times \cdots \times A_n \longrightarrow \Omega.$$

The subobject $[\varphi] \rightarrowtail A_1 \times \cdots \times A_n$ classified by this "truth table" morphism will also be written, in a more suggestive way as

$$\big\{(a_1, \ldots, a_n) \big| \varphi(a_1, \ldots, a_n)\big\} \rightarrowtail A_1 \times \cdots \times A_n.$$

Of course, this notation will be coherent with its corresponding occurrence in Definition 4.2.

Again the definition proceeds inductively, using without recalling them the notation and numeration of Definition 4.3.

Definition 4.5 *In a topos $\mathcal{E}$, with the notation and the numeration of Definition 4.3 and using the various constructions in Definition 2.3, Definition 2.5, Theorem 2.23, Definition 2.12, the* truth table $\ulcorner\varphi\urcorner$ *of a formula φ is defined inductively by:*

1. $\ulcorner \mathsf{true} \urcorner = t$ *and* $\ulcorner \mathsf{false} \urcorner = f$*;*
2. $\ulcorner \tau = \sigma \urcorner = (=_A) \circ (\ulcorner\tau\urcorner, \ulcorner\sigma\urcorner)$*;*
3. $\ulcorner \tau \in \Sigma \urcorner = \in_A \circ (\ulcorner\tau\urcorner, \ulcorner\Sigma\urcorner)$*;*
4. $\ulcorner \neg\varphi \urcorner = \neg \circ \ulcorner\varphi\urcorner$*;*
5. $\ulcorner \varphi \wedge \psi \urcorner = \wedge \circ (\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$,
   $\ulcorner \varphi \vee \psi \urcorner = \vee \circ (\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$,
   $\ulcorner \varphi \Rightarrow \psi \urcorner = (\Rightarrow) \circ (\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$*;*
6. *in 4.3.6, consider the projection $p \colon A \times B_1 \times \cdots \times B_n \longrightarrow B_1 \times \cdots \times B_n$ and the subobject $[\varphi] \rightarrowtail A \times B_1 \times \cdots \times B_n$ classified by $\ulcorner\varphi\urcorner$;*

   $$\ulcorner \exists a \; \varphi(a, b_1, \ldots, b_n) \urcorner, \quad \ulcorner \forall a \; \varphi(a, b_1, \ldots, b_n) \urcorner$$

   are the characteristic morphisms of, respectively, $\exists_p[\varphi]$ and $\forall_p[\varphi]$;
7. $\ulcorner \varphi(\sigma_1, \ldots, \sigma_n) \urcorner = \ulcorner\varphi\urcorner \circ (\ulcorner\sigma_1\urcorner, \ldots, \ulcorner\sigma_n\urcorner)$.

Of course true and false are formulæ without free variable. If you view them as formulæ with a (non appearing) free variable a of type A, the corresponding truth table are then written simply

$$\mathsf{true} \colon A \longrightarrow \mathbf{1} \xrightarrow{\;t\;} \Omega, \quad \mathsf{false} \colon A \longrightarrow \mathbf{1} \xrightarrow{\;f\;} \Omega,$$

or true_A, false_A if some confusion could occur.

Let us conclude this section with an obvious but useful observation.

Proposition 4.6 *In a topos $\mathcal{E}$, consider two formulæ φ, ψ with the same free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$. Write $[\varphi]$ and $[\psi]$ for the subobjects of $A_1 \times \cdots \times A_n$ classified by $\ulcorner\varphi\urcorner$ and $\ulcorner\psi\urcorner$. Then the subobjects of $A_1 \times \cdots \times A_n$ classified by*

$$\ulcorner \mathsf{true} \urcorner, \quad \ulcorner \mathsf{false} \urcorner, \quad \ulcorner \varphi \wedge \psi \urcorner, \quad \ulcorner \varphi \vee \psi \urcorner, \quad \ulcorner \varphi \Rightarrow \psi \urcorner, \quad \ulcorner \neg\varphi \urcorner$$

are simply

$$A_1 \times \cdots \times A_n, \quad 0, \quad [\varphi] \wedge [\psi], \quad [\varphi] \vee [\psi], \quad [\varphi] \Rightarrow [\psi], \quad \neg[\varphi]$$

in the Heyting algebra of subobjects of $A_1 \times \cdots \times A_n$.

Sketch of proof Immediate from the definitions. □

4.3 Propositional calculus in a topos

We have now to explain what it means for a formula to be *true* and to infer the corresponding rules valid in the internal logic of a topos.

Definition 4.7 *In a topos $\mathcal{E}$, let φ be a formula with variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$. We shall say that this formula is* true *and we shall write $\models \varphi$ when $\ulcorner\varphi\urcorner = \ulcorner\mathsf{true}\urcorner$, that is, equivalently, when the subobject classified by $\ulcorner\varphi\urcorner$ is $A_1 \times \cdots \times A_n$ itself.*

Theorem 4.8 *In a topos $\mathcal{E}$, all the axioms of intuitionistic propositional calculus hold. That is, given three formulæ φ, ψ, θ with the same free variables, the following properties hold*

$$
\begin{array}{rl}
(P1) & \models \varphi \Rightarrow (\psi \Rightarrow \varphi) \\
(P2) & \models \big(\varphi \Rightarrow (\psi \Rightarrow \theta)\big) \Rightarrow \big((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \theta)\big) \\
(P3) & \models \varphi \Rightarrow \big(\psi \Rightarrow (\varphi \wedge \psi)\big) \\
(P4) & \models \varphi \wedge \psi \Rightarrow \varphi \\
(P5) & \models \varphi \wedge \psi \Rightarrow \psi \\
(P6) & \models \varphi \Rightarrow (\varphi \vee \psi) \\
(P7) & \models \psi \Rightarrow (\varphi \vee \psi) \\
(P8) & \models (\varphi \Rightarrow \theta) \Rightarrow \big((\psi \Rightarrow \theta) \Rightarrow ((\varphi \vee \psi) \Rightarrow \theta)\big) \\
(P9) & \models (\varphi \Rightarrow \psi) \Rightarrow \big((\varphi \Rightarrow \neg\psi) \Rightarrow \neg\varphi\big) \\
(P10) & \models \neg\varphi \Rightarrow (\varphi \Rightarrow \psi) \\
(P11) & \textit{If} \models \varphi \textit{ and } \models \varphi \Rightarrow \psi \textit{ then } \models \psi \textit{ (Modus Ponens)}
\end{array}
$$

Sketch of proof Assume that our formulæ ϕ, ψ, θ have free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$. By Definition 4.7 and Proposition 4.6, it suffices to prove the corresponding properties in the Heyting algebra of subobjects of $A_1 \times \cdots \times A_n$. We must thus prove that the subobject of $A_1 \times \cdots \times A_n$ classified by the truth value of each of these formulæ is $A_1 \times \cdots \times A_n$ itself.

In fact, *(P1)* to *(P11)* hold in every Heyting algebra (see Definition 2.17). For example if a, b are two elements in a Heyting algebra H, proving *(P1)* is proving

$$1 = \big(a \Rightarrow (b \Rightarrow a)\big).$$

Of course this is equivalent to proving

$$1 \leq \big(a \Rightarrow (b \Rightarrow a)\big).$$

that is

$$a = 1 \wedge a \leq b \Rightarrow a.$$

This is further equivalent to proving

$$a \wedge b \leq a$$

which is obvious. □

The *Modus Ponens* is a so-called *deduction rule* between valid formulæ, not a valid formula for itself.

See Section 6.7 in [4] for a long list of valid formulæ and deduction rules that you can infer from 4.8.

The following proposition underlines a first significant difference with classical logic; see also Example 4.11.

Proposition 4.9 *In a topos $\mathcal{E}$, let φ and ψ be two formulæ with the same free variables.*

- *If* $\models \varphi \wedge \psi$, *then* $\models \varphi$ and $\models \psi$.
- *But if* $\models \varphi \vee \psi$, *one does not have in general* $\models \varphi$ or $\models \psi$.

Sketch of proof As in the proof of Proposition 4.8, it suffices to consider the situation in an arbitrary Heyting algebra H. Given $a, b \in H$, of course $a \wedge b = 1$ forces $a = 1$ and $b = 1$. But trivially also, $a \vee b = 1$ does not imply that one of the two elements is equal to 1. □

But of course there is another major difference with classical logic: the so-called *law of excluded middle*

$$\models \varphi \vee \neg\varphi$$

does not hold in the internal logic of a topos. Going back to Section 3.3, the law of excluded middle holds precisely when the topos is Boolean.

4.4 Predicate calculus in a topos

We consider now the additional logical rules involving quantifiers.

Theorem 4.10 *In a topos $\mathcal{E}$, all the axioms of intuitionistic predicate calculus hold. More explicitly, consider two formulæ φ, ψ with the same free variables. Consider further a term τ. The following properties hold.*

$(P12)$ $\models \big(\forall x(\varphi \Rightarrow \psi)\big) \Rightarrow \big((\forall x\, \varphi) \Rightarrow (\forall x\, \psi)\big)$
$(P13)$ $\models \big(\forall x(\varphi \Rightarrow \psi)\big) \Rightarrow \big((\exists x\, \varphi) \Rightarrow (\exists x\, \psi)\big)$
$(P14)$ $\models \varphi \Rightarrow (\forall x\, \varphi)$ *when x is not a free variable of φ*
$(P15)$ $\models (\exists x\, \varphi) \Rightarrow \varphi$ *when x is not a free variable of φ*
$(P16)$ $\models (\forall x\, \varphi) \Rightarrow \varphi(\tau)$ *when τ does not contain any bound variable of φ and $\varphi(\tau)$ is the result of replacing x by τ in φ*
$(P17)$ $\models \varphi(\tau) \Rightarrow (\exists x\, \varphi)$ *when τ does not contain any bound variable of φ and $\varphi(\tau)$ is the result of replacing x by τ in φ*
$(P18)$ *If* $\models \varphi$ *then* $\models \forall x\, \varphi$

Sketch of proof To simplify the notation, imagine that φ and ψ have the free variables x, a of respective types X, A. The formula in *(P12)* has the free variable a, thus proving *(T12)* reduces to proving the corresponding result in the Heyting algebra of subobjects of A. We consider the projection $p\colon X \times A \longrightarrow A$. Going back to Theorem 2.9, we must prove that

$$A = \big[\big(\forall x(\varphi \Rightarrow \psi)\big) \Rightarrow \big((\forall x\, \varphi) \Rightarrow (\forall x\, \psi)\big)\big]$$

which is the same as

$$A \leq \big[\forall x(\varphi \Rightarrow \psi)\big] \Rightarrow \big(\big[\forall x\, \varphi\big] \Rightarrow \big[\forall x\, \psi\big]\big)$$

By definition of $\Rightarrow$ in a Heyting algebra, this reduces to

$$\big[\forall x(\varphi \Rightarrow \psi)\big] = A \wedge \big[\forall x(\varphi \Rightarrow \psi)\big] \leq \big(\big[\forall x\, \varphi\big] \Rightarrow \big[\forall x\, \psi\big]\big)$$

and further to

$$\big[\forall x(\varphi \Rightarrow \psi)\big] \wedge \big[\forall x\, \varphi\big] \leq \big[\forall x\, \psi\big].$$

Now using the adjunction $p^{-1} \dashv \pi_p$ of Theorem 2.9, this is still equivalent to

$$p^{-1}\Big(\big[\forall x(\varphi \Rightarrow \psi)\big] \wedge \big[\forall x\,\varphi\big]\Big) \leq \big[\psi\big]$$

and thus to

$$p^{-1}\big[\forall x(\varphi \Rightarrow \psi)\big] \wedge p^{-1}\big[\forall x\,\varphi\big] \leq \big[\psi\big].$$

The counit of the adjunction $p^{-1} \dashv \pi_p$ indicates that $p^{-1}\forall x(S) \leq S$ for every subobject S of A. Therefore

$$p^{-1}\big[\forall x(\varphi \Rightarrow \psi)\big] \wedge p^{-1}\big[\forall x\,\varphi\big] \leq [\varphi \Rightarrow \psi] \wedge [\varphi] = \big([\varphi] \Rightarrow [\psi]\big) \wedge [\varphi] \leq [\psi].$$

The other properties are proved in an analogous way. □

Again, see Section 6.8 in [4] for a long list of valid formuæ which can be inferred from Theorem 4.10.

Let us now give an example illustrating one of the differences with classical logic, as mentioned in Proposition 4.9.

Example 4.11 *Consider the topos of sheaves on the real line. The sheaf of continuous functions (see 1.2) satisfies the statement*

$$\models (\exists g\ f \times g = 1) \vee (\exists g\ (1 - f) \times g = 1)$$

but none of the two statements

$$\exists g\ f \times g = 1, \quad \exists g\ (1 - f) \times g = 1$$

is valid.

Sketch of proof Let us write down the proof in the case of a constant f, that is, a continuous function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$. The case of a variable is analogous.

If $f(r) \neq 0$ for some $r \in \mathbb{R}$, then $f(r') \neq 0$ on an open neighborhood U_r of r. Let U be the union of all these U_r. Then $f(r') \neq 0$ for each $r' \in U$, thus on U, it suffices to define $g = \frac{1}{f}$.

Analogously if $f(r) \neq 1$ for some $r \in \mathbb{R}$, then $f(r') \neq 1$ on an open neighborhood V_r of r. Let V be the union of all these V_r. Then $f(r') \neq 1$ for each $r' \in V$, thus on V, it suffices to define $g = \frac{1}{1-f}$.

Trivially, $U \cup V = \mathbb{R}$, which forces the validity of the indicated statement. But if you choose f such that $f(r_1) = 0$ and $f(r_2) = 1$, the statement $\exists g\ f \times g = 1$ is not valid at the neighborhood of r_1 and the statement $\exists g\ (1 - f) \times g = 1$ is not valid at the neighborhood of r_2, thus none of them is a valid statement. □

This example tells us that, in the internal logic of the topos of sheaves on the reals, the sheaf of continuous functions is a "local ring": every element f is invertible, or $1 - f$ is invertible. This example shows also the "local character" of the existential quantifier: the element g exists at the neighborhood of each point, but in general no such global g can be found.

4.5 Structure of a topos in its internal language

Where are we, up to now, in our "scrabble" game? We know how to construct valid terms and formulæ; we have associated with each of these an actual arrow or object of the topos; and we have told what it means for a formula to be true. All right, now in our scrabble game, let me decide that you will win a Belgian chocolate each time that you exhibit a true formula! Even if you like Belgian chocolate, I am afraid that you will not be very anxious to play that game for long. What you are interested in, is to prove theorems about categorical notions in the topos: products, coequalizers, exponentiation, and so on. You are not interested in playing formally with the formal rules of the previous sections ... except maybe, if they can make your life easier in proving actual categorical properties of the topos. To be able to do so, using our "scrabble" game, it remains to translate these categorical notions in terms of the internal logic of the topos. Let us do this for a representative, but non exhaustive, selection of categorical notions.

Proposition 4.12 *In a topos $\mathcal{E}$, consider*

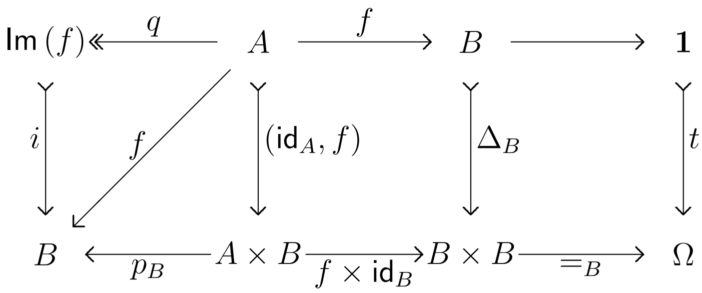
- *objects A, B;*
- *morphisms $f, g\colon A \rightrightarrows B$ and $h\colon C \longrightarrow B$;*
- *subobjects $A_1 \rightarrowtail A$, $A_2 \rightarrowtail A$ and $B' \rightarrowtail B$;*
- *variables a, a' of type A, b of type B and c of type C.*

The following properties hold:

1. $f = g$ iff $\models f(a) = g(a)$;
2. f *is a monomorphism* iff $\models \big(f(a) = f(a')\big) \Rightarrow \big(a = a'\big)$;
3. f *is an epimorphism* iff $\models \exists a\, f(a) = b$;
4. $A_1 \cap A_2 = \{a | (a \in A_1) \wedge (a \in A_2)\}$;
5. $A_1 \cup A_2 = \{a | (a \in A_1) \vee (a \in A_2)\}$;
6. $\mathsf{Im}\, \Sigma_f(A_1) = \big\{b \big| \exists a \,\big((a \in A_1) \wedge (b = f(a)\big)\big\}$;
7. $\pi_f(A_1) = \big\{b \big| \forall a \,\big((f(a) = b) \Rightarrow (a \in A_1)\big)\big\}$;
8. $f^{-1}(B') = \{a | f(a) \in B'\}$;
9. $\mathsf{Im}\,(f) = \{b | \exists a\, f(a) = b\}$;
10. $\mathsf{Ker}\,(f, g) = \{a | f(a) = g(a)\}$;
11. $f \times_B h = \{(a, c) | f(a) = h(c)\}$ *(pullback of f and h).*

Sketch of proof Let us prove the first statement. By definition of the morphism $=$, the subobject $[f(a) = g(a)]$ is the inverse image of the diagonal of B along (f, g), that is the equalizer $\mathsf{Ker}\,(f, g)$. And of course $\mathsf{Ker}\,(f, g) = A$ is equivalent to $f = g$.

The case of epimorphisms is also worth being emphasized. By definition of the existential quantifier, we have a commutative diagram



Both the middle and the right hand square are pullbacks, thus

$$(\mathsf{id}_A, f)\colon A \rightarrowtail A\times B \ \text{ is } \ \{a|f(a)=b\} \cong \{(a,b)|f(a)=b\} \rightarrowtail A\times B.$$

By definition of the existential quantifier

$$\{b|\exists a\ f(a)=b\} = \mathsf{Im}\left(p_B\circ(\mathsf{id}_A,f)\right) = \mathsf{Im}\, f.$$

And of course, $\mathsf{Im}\,(f) = B$ if and only if f is an epimorphism.

See Proposition 6.10.2 in [4] for the other statements. □

But the case of epimorphisms is still worth an additional comment.

Proposition 4.13 *In the topos of sheaves on a locale L, a morphism $f\colon A\longrightarrow B$ is an epimorphism if and only if, for every $u\in L$ and $b\in B(u)$, there exist a covering $u=\bigvee_{i\in I} u_i$ and elements $a_i\in A(u_i)$, such that for every $i\in I$, $f_{u_i}(a_i) = b|_{u_i}$.*

Sketch of proof With the notation of the statement, given $g,h\colon B \rightrightarrows C$ such that $gf=hf$, then g and h coincide on all b_{u_i}, thus also on their gluing b, by the sheaf condition in C.

Conversely, if f is an epimorphism in $\mathsf{Sh}(L)$, consider its image factorization ip in the topos $\mathsf{Pr}(L)$ of presheaves. Applying the associated sheaf functor a to that factorization shows that $B\cong a(I)$ is the sheaf associated with I. Going back to the considerations in Section 3.2, one infers further that B is the closure of I for the Lawvere-Tierney topology defining sheaves. One concludes by Example 3.8. □

Counterexample 4.14 *There exists a localic topos which has an epimorphism which is not pointwise surjective.*

Sketch of proof Consider the topological space X given by the three point set $\{x,y,z\}$ with $\{x,y\}$, $\{y,z\}$ and $\{y\}$ open. With obvious notation, put

$$A\big(\{x,y,z\}\big)=\emptyset,\quad A\big(\{x,y\}\big)=\{a\},\quad A\big(\{y,z\}\big)=\{a'\},\quad A\big(\{y\}\big)=\{a,a'\}.$$

This is a sheaf on X because no gluing condition at all occurs. By Proposition 4.13, the morphism $A\longrightarrow\mathbf{1}$ is an epimorphism in the topos of sheaves, but is not surjective at the top level. □

Analogous considerations could be made in the case of an arbitrary Grothendieck topos. All this shows that the existential quantifier, in the internal logic of a topos, does not at all translate as a pointwise existence in the case of localic or Grothendieck toposes: it translates as a "local" existence, an existence on all the pieces of a covering, or equivalently said in the case of a spatial topos, the existence at the neighborhood of each point. But in an arbitrary topos, when we develop proofs in the internal logic of a topos, Proposition 4.12 tells us that we can nevertheless handle epimorphisms just via the usual formula describing surjections. This is a big simplification, especially if you have to handle combinations of various epimorphisms.

Suggestion(s) for further reading

Here is another occurrence of an interesting property, in the spirit of Example 4.11, which can occur when switching to the internal logic of a topos. Every ring is a local ring ... when considered in an adequate topos! I recall that a local ring is one where for every element r, this element r is invertible or $1 - r$ is invertible. For example, every field is a local ring.

Given a commutative ring R, there exists, in the topos of sheaves on its Zariski spectrum, a ring $\widehat{R}$, which is a local ring in the internal logic of the topos, and admits the original ring R as ring of constants. See Section 2.11 in [4].

4.6 Internal limits and colimits

Many people, who are used to working in Grothendieck toposes, consider that elementary toposes constitute a too poor structure to be useful, since they do not allow handling those deep results which require infinite limits and colimits. Those people are just wrong. First what they do not realize, is that a Grothendieck topos has much more than just limits and colimits of diagrams constituted of a set of objects and a set of arrows: they admit limits and colimits of internal diagrams, constituted of a sheaf of objects and a sheaf of arrows. The consideration of these more involved and general internal limits and colimits can provide serious simplification and elegance in the proofs. And these internal limits and colimits exist in every elementary topos.

Well indeed, an elementary topos is only finitely complete and finitely cocomplete. Thus arbitrary intersections or unions of subobjects, or arbitrary small limits or colimits, do not exist in general. But the internal logic of a topos allows somehow handling "arbitrary internal constructions", without requiring any finiteness condition.

For example, the lattice of subobjects of an object A is a Heyting algebra, but not a locale, because arbitrary unions of subobjects do not exist. As we have seen, we have more: the object Ω^A of the topos, the "object of subobjects of A", is itself an internal Heyting algebra, and the actual subobjects are just the constants of that internal Heyting algebra Ω^A. But in fact, we have even more:

Proposition 4.15 *In an elementary topos $\mathcal{E}$, for every object A, the Heyting algebra Ω^A is provided with the structure of an internal locale.*

Sketch of proof As just recalled, Ω^A should be thought as the "object of subobjects" of A. Thus a subobject $S \rightarrowtail \Omega^A$, that is, a constant of type Ω^{Ω^A}, can be thought as a family of subobjects of A. The following expressions make sense in the internal logic of the topos, with a a variable of type A an σ a variable of type Ω^A, and they define actual subobjects of A:

$$\bigcap S = \{a \mid \forall \sigma\, (\sigma \in S \Rightarrow a \in \sigma)\} \subseteq A$$

$$\bigcup S = \{a \mid \exists \sigma\, (\sigma \in S \wedge a \in \sigma)\} \subseteq A$$

These subobjects are of course the internal intersection and the internal union of the internal family of internal subobjects. Refining a little bit the same argument and switching to a variable S of type Ω^{Ω^A}, instead of a constant, yields now morphisms

$$\bigcup : \Omega^{\Omega^A} \longrightarrow \Omega^A$$

$$\bigcap : \Omega^{\Omega^A} \longrightarrow \Omega^A$$

with $\bigcup$ expressing the internal locale structure of Ω^A. Indeed this morphism must correspond by Cartesian closedness to a morphism

$$A \times \Omega^{\Omega^A} \longrightarrow \Omega$$

thus to a subobject of $A \times \Omega^{\Omega^A}$, namely,

$$\big\{(a, S)\big|\exists\sigma \ \ (\sigma \in S \wedge a \in \sigma)\big\}.$$

See Theorem 6.1.9 in [4] for the details. □

Now as far as limits and colimits are concerned. Given a category $\mathcal{E}$ and a small category $\mathcal{D}$, the existence of of all $\mathcal{D}$-limits or $\mathcal{D}$-colimits is equivalent to the existence of a right or left adjoint to the functor

$$\Delta_{\mathcal{D}} \colon \mathcal{E} \longrightarrow [\mathcal{D}, \mathcal{E}], \quad X \mapsto \Delta_X$$

where Δ_X indicates the constant functor on X (see Proposition 3.2.3 in [3]).

Still with $\mathcal{D}$ a small category; write $\mathsf{Ob}(\mathcal{D})$ and $\mathsf{Ar}(\mathcal{D})$ for its sets of objects and morphisms. A functor $F \colon \mathcal{D} \longrightarrow \mathsf{Set}$ consists in giving a family $\big(F(D)\big)_{D \in \mathsf{Ob}(\mathcal{D})}$ of sets, together with an action of $\mathsf{Ar}(\mathcal{D})$ on this family. But as we know, this can be rephrased as a morphism $\coprod_{D \in \mathsf{Ob}(\mathcal{D})} F(D) \longrightarrow \mathsf{Ob}(\mathcal{D})$, provided thus with an action of $\mathsf{Ar}(\mathcal{D})$.

Given now an internal category $\mathbb{D}$ in a topos $\mathcal{E}$ (see Definition 8.1.1 in [2]), with D_0 as object of objects and D_1 as object of morphisms, an internal base valued functor $F \colon \mathbb{D} \longrightarrow \mathcal{E}$ is defined as a pair (F, ϕ), with $\phi \colon F \longrightarrow D_0$, together with an action of D_1 on F (see Definition 8.2.1 in [2]). Given an object $X \in \mathcal{E}$, the constant internal functor on X is thus the projection $D_0 \times X \longrightarrow D_0$, with the identity action of D_1. It is straightforward to define internal natural transformations and get so an actual functor

$$\Delta_{\mathbb{D}} \colon \mathcal{E} \longrightarrow [\mathbb{D}, \mathcal{E}]$$

to the category of internal base valued functors. (see Section 8.3 in [3])

Proposition 4.16 *An elementary topos $\mathcal{E}$ is internally complete and cococomplete, meaning that each functor $\Delta_{\mathbb{D}}$ as above has both a right and a left adjoint.*

Sketch of proof It is an excellent exercise to prove that the classical formulæ describing limits and colimits in Set, when interpreted in the internal logic of the elementary topos $\mathcal{E}$, yield the expected result. □

Now given a site $(\mathcal{C}, \mathcal{T})$, the composite

$$\mathsf{Set} \xrightarrow{\ \Delta_{\mathcal{C}}\ } \mathrm{Pr}(\mathcal{C}) \xrightarrow{\ a\ } \mathsf{Sh}(\mathcal{C}, \mathcal{T}),$$

with a the associated sheaf functor, preserves finite limits since both functors do. It transforms thus every small category $\mathcal{D}$ – every internal category in Set – in an internal category $\mathbb{D}$ in $\mathsf{Sh}(\mathcal{C}; \mathcal{T})$. A rather trivial internal category in fact, because

$$D_0 = a\Delta\big(\mathsf{Ob}(\mathcal{D})\big) = \coprod_{\mathsf{Ob}(\mathcal{D})} \mathbf{1}, \quad D_1 = a\Delta\big(\mathsf{Ar}(\mathcal{D})\big) = \coprod_{\mathsf{Ar}(\mathcal{D})} \mathbf{1}$$

because it is the case in Set, while Δ and a preserve coproducts, since they have a right adjoint.

Moreover a functor $F\colon \mathcal{D} \longrightarrow \mathsf{Sh}(\mathcal{C}, \mathcal{T})$ yields at once an internal based valued functor

$$\coprod_{D \in \mathcal{D}} F(D) \xrightarrow{\coprod \xi_D} \coprod_{\mathsf{Ob}(\mathcal{D})} \mathbf{1} \cong D_0$$

where $\xi_D\colon F(D) \longrightarrow \mathbf{1}$. It is routine to observe that the internal limit or colimit of this internal based valued functor recaptures the ordinary small limit or colimit of F.

Thus in a Grothendieck topos, small limits and colimits are just particular cases of internal limits and colimits. Of course, in a Grothendieck topos, internal completeness and cocompleteness are much richer properties than just small completeness and cocompleteness. And as Proposition 4.16 attests, these richer porperties are valid in every elementary topos.

Lesson 5

Classifying toposes

Very, very roughly speaking ... What is a classifying topos?

Consider a mathematical theory $\mathcal{T}$ for which it makes sense to speak of the models of $\mathcal{T}$ in a Grothendieck topos. For example rings, Heyting algebras, ordered groups, modules, and so on. We say that this theory admits a classifying topos when, somewhere in the nature, there exist a Grothendieck topos $\mathcal{E}[\mathcal{T}]$ canonically associated with $\mathcal{T}$ and, in this topos $\mathcal{E}[\mathcal{T}]$, a model M of our theory $\mathcal{T}$, a model which we call the "generic model of $\mathcal{T}$". Why generic? Because all the models of $\mathcal{T}$ in all Grothendieck toposes are exactly the images of that generic model M by all possible morphisms of toposes.

Thus, we should start with studying the morphisms of toposes. And of course, a necessary condition for a theory to admit a classifying topos will be that its models are preserved by the morphisms of toposes.

5.1 A quick review of Kan extensions

This section recalls some basic facts about Kan extensions, a notion which is probably less widely known than limits and adjoint functors. In this context, the category of elements of a set-valued functor appears as a very useful tool.

Definition 5.1 *Given a functor $F\colon \mathcal{A}\longrightarrow\mathsf{Set}$, its category $\mathsf{Elt}(F)$ of elements has for objects the pairs (A,a), with A an object of $\mathcal{A}$ and $a \in F(A)$. A morphism $f\colon (A,a)\longrightarrow(B,b)$ is a morphism $f\colon A\longrightarrow B$ in $\mathcal{A}$ such that $F(f)(a)=b$. We shall write $\phi_F\colon \mathsf{Elt}(F)\longrightarrow\mathcal{A}$ for the obvious forgetful functor.*

Of course when the functor F is contravariant, the definition of morphisms in $\mathsf{Elt}(F)$ involves the equality $F(f)(b)=a$.

Proposition 5.2 *Every functor $F\colon \mathcal{A}\longrightarrow\mathsf{Set}$, with $\mathcal{A}$ a small category, is a colimit of representable functors.*

Sketch of proof F is the colimit of the composite

$$\mathsf{Elt}(F)^{op}\xrightarrow{\ \phi_F^{op}\ }\mathcal{A}^{op}\xrightarrow{\ Y^*\ }\mathsf{Funct}[\mathcal{A},\mathsf{Set}]$$

where $Y^*(A)=\mathcal{C}(A,-)$ is the contravariant Yoneda embedding. The canonical morphisms of the colimit

$$\sigma_{(A,a)}\colon \mathcal{A}(A,-)\Rightarrow F$$

are the natural transformations corresponding by the Yoneda lemma to the objects $(A,a)\in \mathsf{Elt}(F)$. $\square$

Observe that in Proposition 5.2, the Yoneda embedding is a contravariant functor.

Proposition 5.3 *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be categories, with $\mathcal{A}$ small. Given a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$, consider the functor "composition with F"*

$$- \circ F\colon [\mathcal{B}, \mathcal{C}] \longrightarrow [\mathcal{A}, \mathcal{C}], \quad G \mapsto G \circ F.$$

- *When $\mathcal{C}$ is cocomplete, this functor admits a left adjoint written Lan_F; given $H \in [\mathcal{A}, \mathcal{C}]$, the functor $\mathsf{Lan}_F(H)$ is called the left Kan extension of H along F.*
- *When $\mathcal{C}$ is complete, this functor admits a right adjoint written Ran_F; given $H \in [\mathcal{A}, \mathcal{C}]$, the functor $\mathsf{Ran}_F(H)$ is called the right Kan extension of H along F.*

Sketch of proof We shall only need the case of Lan_F; see Theorem 3.7.2 in [2]. Given $B \in \mathcal{B}$, consider the functor $\mathcal{B}\big(F(-), B\big): \mathcal{A} \longrightarrow \mathsf{Set}$. $\mathsf{Lan}_F(H)(B)$ is the colimit of the composite

$$\mathsf{Elt}\Big(\mathcal{B}\big(F(-), B\big)\Big) \xrightarrow{\phi_{\mathcal{B}(F(-),B))}} \mathcal{A} \xrightarrow{\quad H \quad} \mathcal{C}.$$

that is

$$\mathsf{Lan}_F(H)(B) = \mathsf{colim}_{\, f\colon F(A) \to B} H(A).$$ □

Proposition 5.4 *In the conditions of Proposition 5.3, when the functor F is full and faithful, the triangles*

are commutative, up to an isomorphism.

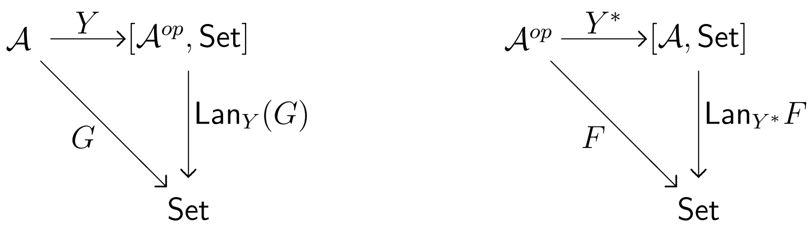
Sketch of proof See Proposition 3.7.3 in [2]. Given $A \in \mathcal{A}$, the pair $(A, \mathsf{id}_{F(A)})$ is now a terminal object in the category $\mathsf{Elt}\Big(\mathcal{B}\big(F(-), F(A)\big)\Big)$, thus the colimit involved is the image of that terminal object (see Corollary 2.11.5 in [2]),which is thus $H(A)$. □

The next result will be essential in the rest of this lesson.

Proposition 5.5 *Consider two functors*

$$F\colon \mathcal{A}^{op} \longrightarrow \mathsf{Set}, \quad G\colon \mathcal{A} \longrightarrow \mathsf{Set}$$

defined on a small category $\mathcal{A}$. Consider further



$$Y(A) = \mathcal{A}(-, A) \qquad\qquad Y^*(A) = \mathcal{A}(A, -)$$

where Y and Y^* are the Yoneda embeddings. In those conditions

$$\mathsf{Lan}_Y G(F) \cong \mathsf{Lan}_{Y^*} F(G)$$

and that set is generally denoted[1] $F \otimes G$.

Sketch of proof See Proposition 3.8.1 in [2]. Writing F and G as colimits of representable functors (see Propositon 5.2), and using the colimit formulæ of Proposition 5.3, this is essentially due to the commutation of colimits with colimits. □

5.2 The case of left exact functors

We now switch to Kan extensions of left exact functors.

Proposition 5.6 *Let $\mathcal{A}$ be a small category with finite limits. A functor $F\colon \mathcal{A} \longrightarrow \mathsf{Set}$ is left exact if and only if its category of elements is cofiltered*[2]*, that is, if and only if F is a filtered colimit of representable functors*

Sketch of proof The functor $Y^* \circ \phi_F$ in Proposition 5.2 is contravariant, thus when the category $\mathsf{Elt}(F)$ is cofiltered, the colimit of $Y \circ \phi_F$ is filtered. In that case, since each representable functor preserves finite limits and finite limits commute in Set with filtered colimits (see Proposition 1.22.1), the colimit F preserves finite limits. Conversely when $\mathcal{A}$ has finite limits preserved by F, it is obvious that $\mathsf{Elt}(F)$ is cofiltered.

See Propositions 6.3.2 and 6.3.7 in [2] for more details. □

Proposition 5.6 can further be used pointwise to characterize left exact functors with values in Grothendieck toposes:

Proposition 5.7 *A functor $F\colon \mathcal{A} \longrightarrow \mathcal{E}$ to a Grothendieck topos $\mathcal{E} = \mathsf{Sh}(\mathcal{C}, \mathcal{S})$ is left exact if and only if each composite*

$$\mathcal{A} \xrightarrow{F} \mathcal{E} = \mathsf{Sh}(\mathcal{C}, \mathcal{S}) \xrightarrow{\mathsf{ev}_C} \mathsf{Set}$$

with the evaluation functor ev_C, for every $C \in \mathcal{C}$, is left exact.

Sketch of proof In a Grothendieck topos $\mathsf{Sh}(\mathcal{C}, \mathcal{S})$, (finite) limits are computed as in $\mathrm{Pr}(\mathcal{C})$, where they are pointwise. Thus a functor is left exact when it is pointwise left exact. □

The following characterization of left exact functors will be essential for our purpose.

Proposition 5.8 *Let $\mathcal{A}$ be a small category with finite limits and $F\colon \mathcal{A} \longrightarrow \mathsf{Set}$ a functor. The following conditions are equivalent:*

1. *F is left exact;*

2. *the left Kan extension $\mathsf{Lan}_Y F$ of F along the covariant Yoneda embedding is left exact.*

Sketch of proof We have the situation

[1] The additive version of this result, with $\mathcal{A}$ a ring viewed as a one-object additive category, yields precisely the definition of the tensor product of a right and a left module, from where the notation.

[2] A category is cofiltered when it there exists a cone on each finite diagram in it, and filtered when there exists a cocone on each finite diagram.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{Y} & \Pr(\mathcal{A}) \\ & {\scriptstyle F}\searrow & \downarrow {\scriptstyle \mathsf{Lan}_Y F} \\ & & \mathsf{Set} \end{array} \qquad Y(A) = \mathcal{A}(-, A)$$

The covariant Yoneda embedding Y preserves limits and the triangle is commutative by Proposition 5.4; thus $(2 \Rightarrow 1)$.

Conversely by Proposition 5.5, for every $G \in \Pr(\mathcal{C})$

$$\mathsf{Lan}_Y(F)(G) = \mathsf{Lan}_{Y^*}(G)(F).$$

Via the Yoneda lemma, the second term is a colimit computed on

$$\mathsf{Elt}\big(\Pr(\mathcal{A})(Y^*(-), F\big) \cong \mathsf{Elt}(F).$$

This is a filtered colimit, because F is left exact. Once more the commutation of finite limits and filtered comimits in Set allows to infer that $\mathsf{Lan}_Y F$ is left exact. $\square$

Suggestion(s) for further reading

A functor $F\colon \mathcal{A} \longrightarrow \mathsf{Set}$ defined on an arbitrary small category $\mathcal{A}$ is *flat* when its category of elements is cofiltered. This is equivalent to the left-Kan extension of F along the covariant Yoneda embedding being left exact. See Chapter 6 in [2] for the theory of flat functors.

5.3 Geometric morphisms

The notion of geometric morphism of toposes is directly inspired from the case of sheaves on topological spaces.

Proposition 5.9 *A continuous mapping $f\colon X \longrightarrow Y$ between topological spaces induces a pair of adjoint functors*

$$f_*\colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(Y), \quad f^*\colon \mathsf{Sh}(Y) \longrightarrow \mathsf{Sh}(X), \quad f^* \dashv f_*$$

with the functor f^* preserving finite limits.

Sketch of proof With the notation of Section 1.1, given a sheaf F on X, the composite

$$\mathcal{O}(Y) \xrightarrow{f^{-1}} \mathcal{O}(X) \xrightarrow{F} \mathsf{Set}$$

is a sheaf on Y which we define to be $f_*(F)$.

One gets the left adjoint $f^*(G)$ by computing first the left Kan extension $\mathsf{Lan}_{f^{-1}}(G)$ of G along f^{-1} and applying next the associated sheaf functor.

The adjunction $f^* \dashv f_*$ follows at once from the two adjunctions involving the left Kan extension and the associated sheaf functor.

To prove that this adjoint f^* preserves finite limits, it suffices to prove that $\mathsf{Lan}_{f^{-1}}$ does, since we know already by Theorem 1.20 that it is the case for the associated sheaf functor.

We know by Proposition 5.3 that $\mathsf{Lan}_{f^{-1}}(G)(U) = \mathsf{colim}\, G \circ \phi_U$ where, for clarity, we write things covariantly:

$$
\begin{array}{ccccc}
\mathsf{Elt}_U & \xrightarrow{\phi_U} & \mathcal{O}(Y)^{op} & \xrightarrow{f^{-1}} & \mathcal{O}(X)^{op} \\
 & & & \searrow^{G} & \downarrow{\scriptstyle \mathsf{Lan}_{f^{-1}}(G)} \\
 & & & & \mathsf{Set}
\end{array}
$$

The category Elt_U has for objects the $V \in \mathcal{O}(Y)$ such that $f^{-1}(V) \supseteq U$, with a morphism $V \longrightarrow W$ when $V \supseteq W$. And of course $\phi_U(V) = V$. The category Elt_U is trivially cofiltered, since $f^{-1}(V) \supseteq U$ and $f^{-1}(W) \supseteq U$ imply $f^{-1}(V \cap W) \supseteq U$. The left exactness of the Kan extension follows then from the commutation in Set of finite limits with filtered colimits. □

The definition in which we are interested is the following one:

Definition 5.10 *A* geometric morphism $f\colon \mathcal{E} \longrightarrow \mathcal{F}$ *of toposes is a pair of adjoint functors*

$$f_*\colon \mathcal{E} \longrightarrow \mathcal{F}, \quad f^*\colon \mathcal{F} \longrightarrow \mathcal{E}, \quad f^* \dashv f_*$$

with the functor f^* preserving finite limits. The functor f_* is called the direct image *functor and the functor f^*, the* inverse image *functor.*

Not all authors agree on the direction of a geometric morphism: that of f_* or that of f^*. The direction of f_* sounds reasonable in view of Proposition 5.9 and we shall adopt it in these notes, as most authors do. But the theory of the classifying topos could possibly plead for the other choice.

Examples 5.11 *We already met the following examples of geometric morphisms.*

1. *given a continuous mapping $f\colon X \longrightarrow Y$ between topological spaces, the corresponding geometric morphism of Proposition 5.9;*

2. *given a topology j on the object Ω of an elementary topos $\mathcal{E}$, the inclusion functor together with its left adjoint, the associated sheaf functor:*

$$i\colon \mathsf{Sh}_j \rightarrowtail \mathcal{E}, \quad a\colon \mathcal{E} \longrightarrow \mathsf{Sh}_j \quad a \dashv i;$$

3. *given a morphism $f\colon I \longrightarrow J$ in an elementary topos $\mathcal{E}$, the functor π_f and its left adjoint f^{-1}*

$$\pi_f\colon \mathcal{E}/I \longrightarrow \mathcal{E}/J, \quad f^{-1}\colon \mathcal{E}/J \longrightarrow \mathcal{E}/I, \quad f^{-1} \dashv \pi_f;$$

4. *given a small category $\mathcal{A}$, the "limit functor" together with its left adjoint, the "constant presheaf functor"*

$$\lim\colon \ \mathrm{Pr}(\mathcal{A}) \longrightarrow \mathsf{Set}, \quad \Delta\colon \mathsf{Set} \longrightarrow \mathrm{Pr}(\mathcal{A}) \quad \Delta \dashv \lim .$$

Sketch of proof In the third example, f^{-1} preserves limits since it admits the left adjoint Σ_f. In the last example, Δ preserves limits since it admits the left adjoint colim. □

Let us also observe that:

Proposition 5.12 *Given a Grothendieck topos $\mathcal{E}$, there exists a unique (up to isomorphism) geometric morphism $f\colon \mathcal{E} \longrightarrow \mathsf{Set}$.*

Sketch of proof An inverse image functor $f^*\colon \mathsf{Set} \longrightarrow \mathcal{E}$ is such that $f^*(\mathbf{1}) = \mathbf{1}$, since it preserves finite limits. It preserves also arbitrary coproducts, thus $f^*(A)$ must be the A-copower of $\mathbf{1}$. This proves the uniqueness. The existence is proved by just composing examples 2 and 4 in 5.11. □

The following notion is also of common use.

Definition 5.13 *By a point of a Grothendieck topos $\mathcal{E}$ is meant a geometric morphism $p\colon \mathsf{Set} \longrightarrow \mathcal{E}$.*

This definition is inspired by:

Proposition 5.14 *Every point $x \in X$ of a topological space X induces a point of the topos $\mathsf{Sh}(X)$ of sheaves on X.*

Sketch of proof Just apply Proposition 5.9 to $\{x\} \rightarrowtail X$. □

Suggestion(s) for further reading

A closed subset of a topological space X is irreducible when it is non empty and cannot be written as the union of two non-empty strictly smaller closed subsets. The space X is *sober* when every irreducible closed subset is the closure of a unique point (see Section 1.9 in [4]). In that case, Proposition 5.14 describes a bijection between the points of X and the points of the topos $\mathsf{Sh}(X)$ (see Definition 1.9.1 and Corollary 2.12.3 in [4]). Every Hausdorff space is sober.

5.4 The classifying topos of a finite limit theory

As a first approach to the study of classifying toposes, let us investigate the case of the theory $\mathcal{T}$ given by a small category $\mathbb{T}$ with finite limits, and whose models in a Grothendieck topos $\mathcal{E}$ are the left exact functors $F\colon \mathbb{T} \longrightarrow \mathcal{E}$. This covers thus the case of algebraic theories (just finite products are needed), but also the theories of small categories, of small groupoids, of preordered sets, and so on.

Theorem 5.15 *Let $\mathbb{T}$ be a small category with finite limits. The theory $\mathcal{T}$ left exact functors on $\mathbb{T}$ admits a classifying topos: this is the topos $\mathrm{Pr}(\mathbb{T})$ of presheaves on $\mathbb{T}$, together with the Yoneda embedding as generic model.*

Sketch of proof See Theorem 4.2.1 in [4] for a detailed proof. The Yoneda embedding $Y\colon \mathbb{T} \longrightarrow \mathrm{Pr}(\mathbb{T})$ preserves (finite) limits, thus is a model in $\mathrm{Pr}(\mathbb{T})$ of the theory $\mathcal{T}$ of left exact functors on $\mathbb{T}$. It is in fact the expected generic model: for every Grothendieck topos $\mathcal{E}$

$$\mathsf{Geom}\big(\mathcal{E}, \mathrm{Pr}(\mathbb{T})\big) \longrightarrow \mathsf{Lex}\big(\mathbb{T}, \mathcal{E}\big), \quad f \mapsto f^* \circ Y$$

is an equivalence of categories.

The proof shows that the inverse equivalence maps a left exact functor F on the geometric morphism g determined by $g^* = \mathsf{Lan}_Y F$, the left Kan extension of F along the Yoneda embedding. This functor g^* is left exact, as follows from the results in Section 5.2.

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\;Y\;} & \mathrm{Pr}(\mathbb{T}) \\ & {\scriptstyle F}\searrow & \downarrow {\scriptstyle \mathsf{Lan}_Y F} \\ & & \mathcal{E} \end{array}$$

The definition of the right adjoint g_* to g^* is then imposed by the Yoneda lemma and the adjunction requirement:

$$g_*(X)(T) \cong \mathsf{Nat}\big(\mathbb{T}(-,T), g_*(X)\big) \cong \mathcal{E}\Big(g^*\big(\mathbb{T}(-,T)\big), X\Big),$$

with thus T an object of $\mathbb{T}$ and X an object of $\mathcal{E}$ $\square$

Suggestion(s) for further reading

A flat functor $F\colon \mathbb{T} \longrightarrow \mathcal{E}$ to a Grothendieck topos is defined as a functor whose left Kan extension $\mathsf{Lan}_Y F$ along the Yoneda embedding is left exact. The theory of flat functors on a small category admits the topos $\Pr(\mathbb{T})$ as classifying topos, with again the Yoneda embedding as generic model. Moreover, every Grothendieck topos is the classifying topos of a theory of flat functors defined on a small category $\mathbb{T}$, which map some specified cocones on colimit cocones. See Section 6.3 in [2] and Proposition 4.3.1 in [4].

5.5 Coherent theories

We describe now a wide class of theories, defined in terms of operations and relations, and whose models are preserved by the inverse image functor f^* of every geometric morphism f. To avoid hiding the spirit of our definitions behind unessential technical details, we omit them and refer to [4] for these details, which concern essentially the sets of variables.

Definition 5.16 *A* coherent theory $\mathcal{T}$ *consists in giving*

- *type symbols;*
- *constants and variables with a prescribed type;*
- *triples* $\big(\tau, (A_1, \ldots, A_n), A\big)$ *where* τ *is an* operation*symbol,* $(A_1, \ldots, A_n)$ *is a finite sequence of types called the* domain *of* τ *and* A *is a type called the* codomain *of* τ*;*
- *pairs* $\big(R, (A_1, \ldots, A_n)\big)$ *where* R *is a* relation *symbol and* $(A_1, \ldots, A_n)$ *is a finite sequence of types, called the* signature *of* R*;*
- *axioms of the form* $\models (\varphi \Rightarrow \psi)$ *where* φ *and* ψ *are coherent formulæ.*

The coherent terms and formulæ are defined inductively, analogously to what has been done in Definitions 4.2 and 4.3:

- *the constants are coherent terms;*
- *when* τ *is an operation with domain* $(A_1, \ldots, A_n)$ *and codomain* A*, while the* σ_i *are coherent terms of respective types* A_i*, then* $\tau(\sigma_1, \ldots, \sigma_n)$ *is a coherent term of type* A*;*
- true *and* false *are coherent formulæ;*
- *if* σ *and* σ' *are coherent terms with the same type,* $\sigma = \sigma'$ *is a coherent formula;*
- *if* φ *and* ψ *are coherent formulæ, then* $\varphi \wedge \psi$ *and* $\varphi \vee \psi$ *are coherent formulæ;*
- *if* φ *is a coherent formula with free variables* $x, a_1, \ldots, a_n$*, then* $\exists x \varphi(x, a_1, \ldots, a_n)$ *is a coherent formula;*

- *when R is a relation of signature $(A_1, \dots, A_n)$ and the σ_i are coherent terms of respective types A_i, then $R(\sigma_1, \dots, \sigma_n)$ is a coherent formula.*

See Definition 6.5.8 in [4] for an exhaustive description taking an explicit care of the sets of variables.

You will have noticed that in the definition of coherent terms and formulæ, the symbols $\neg$, $\Rightarrow$, $\forall$, as well as terms of the form $\{x|\varphi(x)\}$ are not allowed. Going back to the definition of these various ingredients, you will further notice that indeed, they have no reason to be preserved by the inverse image of a geometric morphism.

Definition 5.17 *Let $\mathcal{T}$ be a coherent theory. A* model *of $\mathcal{T}$ in a topos $\mathcal{E}$ consists in first specifying in $\mathcal{E}$*

- *an object $\ulcorner A \urcorner$ for each type of the theory;*
- *a morphism $\mathbf{1} \longrightarrow \ulcorner A \urcorner$ for each constant of type A;*
- *a morphism $\ulcorner A_1 \urcorner \times \cdots \times \ulcorner A_n \urcorner \longrightarrow \ulcorner A \urcorner$ for each operation τ as in Definition 5.16;*
- *a subobject $\ulcorner R \urcorner \rightarrowtail \ulcorner A_1 \urcorner \times \cdots \times \ulcorner A_n \urcorner$ for each relation R as in Definition 152;*
- *given a relation R of signature $(A_1, \dots, A_n)$, the formula $R(a_1, \dots, a_n)$, in the topos $\mathcal{E}$, is defined as being $\exists r\big((a_1, \dots, a_n) = r\big)$, with r a variable of type $\ulcorner R \urcorner$ and each a_i, a variable of type $\ulcorner A_i \urcorner$.*

Those data constitute a model of $\mathcal{T}$ when all the axioms of $\mathcal{T}$ become valid formulæ in the internal logic of the topos $\mathcal{E}$.
A morphism between two such models consists as usual in a family of morphisms of $\mathcal{E}$, one for each type of the theory, in such a way that all the operations and relations of the theory commute with this family of morphisms.
See Definitions 6.5.9 and 6.5.10 in [4] for exhaustive descriptions.

Proposition 5.18 *Let $f \colon \mathcal{E} \longrightarrow \mathcal{F}$ be a geometric morphism of toposes. The inverse image functor $f^* \colon \mathcal{F} \longrightarrow \mathcal{E}$ preserves the models of every coherent theory.*

Sketch of proof Let $\mathcal{T}$ be a coherent theory. Given a geometric morphism f, the inverse image f^* preserves finite limits and colimits, thus in particular operations, relations, coherent terms and formulæ. It preserves also the validity of an axiom $\varphi \Rightarrow \psi$ with φ and ψ coherent, because this reduces to $[\varphi] \subseteq [\psi]$, as the comment concluding Section 2.5 shows. $\Box$

5.6 The classifying topos of a coherent theory

Let us now see that every coherent theory admits a classifying topos.

Lemma 5.19 *Let $\mathbb{T}$ be a small category with finite limits, provided with a specified set $\mathcal{D}$ of discrete cocones. The theory $\mathcal{T}$ of left exact functors on $\mathbb{T}$, which transform every discrete cocone of $\mathcal{D}$ in an epimorphic family, has a classifying topos.*

Sketch of proof Each family $(f_i\colon T_i \longrightarrow T)_{i\in I}$ in $\mathbb{T}$ generates a corresponding sieve

$$r\colon R \rightarrowtail \mathbb{T}(-,T).$$

Let $\mathcal{S}$ be the smallest Grothendieck topology containing all these sieves. The expected classifying topos is the the topos of sheaves $\mathsf{Sh}(\mathbb{T},\mathcal{S})$.

The composite of the Yoneda embeding and the associated sheaf functor

$$\mathbb{T} \xrightarrow{\;Y\;} \mathrm{Pr}(\mathbb{T}) \xrightarrow{\;a\;} \mathsf{Sh}(\mathbb{T},\mathcal{S})$$

is a model of the theory. Indeed given a family $(f_i\colon T_i \longrightarrow T)_{i\in I}$ as above in $\mathbb{T}$, and the corresponding sheaf R, consider the following diagram in $\mathrm{Pr}(\mathbb{T})$

$$\begin{array}{ccccc}
\mathbb{T}(-,T_i) & \xrightarrow{\;\mathbb{T}(-,f_i)\;} & \mathbb{T}(-,T) & & \\
\downarrow{\scriptstyle \eta_{T_i}} & & \downarrow{\scriptstyle \eta_T} & & \\
a\mathbb{T}(-,T_i) & \xrightarrow{\;a\mathbb{T}(-,f_i)\;} & a\mathbb{T}(-,T) & \overset{u}{\underset{v}{\rightrightarrows}} & F
\end{array}$$

where the vertical morphisms are units of the adjunction $a \dashv i$, with a the associated sheaf functor. If F is a sheaf and $u \circ a\mathbb{T}(-,f_i) = v \circ a\mathbb{T}(-,f_i)$ for all i, then $u \circ \eta_T \circ \mathbb{T}(-,f_i) = v \circ \eta_T \circ \mathbb{T}(-,f_i)$ for all i and therefore, $u \circ \eta_T$ and $v \circ \eta_T$ coincide on $R \subseteq \mathbb{T}(-,T)$. Since R is covering, the uniqueness condition in the definition of a sheaf forces $u \circ \eta_T = v \circ \eta_T$. But since η_T is a unit of the adjunction $a \dashv i$ and F is a sheaf, the uniqueness condition in the corresponding universal property forces $u = v$. Thus $a \circ Y$ is a $\mathcal{T}$-model, which we choose as the generic one.

Given a geometric morphism $f\colon \mathcal{E} \longrightarrow \mathsf{Sh}(\mathbb{T},\mathcal{S})$, the inverse image functor f^* is left exact and admits the right adjoint f_*, thus preserves colimits, and therefore epimorphic families. Composing with the generic model yields thus a corresponding $\mathcal{T}$-model in $\mathcal{E}$. Once more, given a $\mathcal{T}$-model in $\mathcal{E}$, one extends it by Kan extension in the inverse image of a geometric morphism.

See Proposition 4.3.8 in [4] for a detailed proof. □

Theorem 5.20 *Every coherent theory $\mathcal{T}$ admits a classifying topos.*

Sketch of proof Once more we refer to Theorem 4.4.1 in [4] for an explicit proof. One constructs first a directed graph[3] $\mathcal{G}$ associated with the coherent theory $\mathcal{T}$. With the notation of Definition 5.16:

- each finite sequence $(A_1,\ldots,A_n)$ of types is chosen as an object in the graph; we add formally in the graph an arrows $p_i\colon (A_1,\ldots,A_n) \longrightarrow A_i$ for each index i;
- for each constant c of type A, we introduce a morphism $c\colon (\,) \longrightarrow A$ in the graph, where $(\,)$ indicates the empty sequence of types;
- for each operation τ with domain $(A_1,\ldots,A_n)$ and codomain A, we introduce a morphism $\tau\colon (A_1,\ldots,A_n) \longrightarrow A$ in the graph;
- for each relation R of signature $(A_1,\ldots,A_n)$, we put further an object R in the graph, together with an arrow $r\colon R \longrightarrow (A_1,\ldots,A_n)$.

[3] A *directed graph* consists in giving just objects and arrows between these, without any further requirement.

We consider next the "path category" $\mathcal{P}$ of the graph $\mathcal{G}$: the objects are those of $\mathcal{G}$ and the morphisms are the finite sequences of "consecutive" arrows in $\mathcal{G}$. Finally, one proves the existence of a small category $\mathbb{T}$, universally associated with $\mathcal{P}$, which is finitely complete, in which each $(A_1, \dots, A_n)$ is now the product $(A_1)\times\cdots\times(A_n)$ and each $r\colon (A_1, \dots, A_n)\longrightarrow(A)$ is now a monomorphism.

Consider the coherent theory $\mathcal{T}_0$ obtained from the given one by keeping all the constants, operations and relations, but dropping all the axioms. The universal construction of the category $\mathbb{T}$ implies that the models of $\mathcal{T}_0$ in a Grothendieck topos $\mathcal{E}$ can equivalently be presented as the left exact functors $\mathbb{T}\longrightarrow\mathcal{E}$.

It remains thus to take care of the axioms of the theory $\mathcal{T}$. To achieve this, for every coherent formula φ with free variables $a_1, \dots, a_n$ of types $A_1, \dots, A_n$, we shall exhibit a finite family of morphisms

$$\big(f_i\colon B_i \longrightarrow A_1 \times \cdots \times A_n\big)_{i=1,\dots,k}$$

in $\mathbb{T}$ such that, for each left exact functor $F\colon \mathcal{C}\longrightarrow\mathcal{E}$ to a Grothendieck topos $\mathcal{E}$,

$$[\varphi] = \mathsf{Im}\, F(f_1) \cup \cdots \cup F(f_k) \rightarrowtail F(A_1) \times F(A_n)$$

where $[\varphi]$ is the subobject classified by the truth table of φ (see Section 4.2). This will be done by induction on the complexity of the coherent formula φ. The validity of the formula φ in the topos $\mathcal{E}$, that is, $[\varphi] = F(A_1) \times \cdots \times F(A_n)$ will thus precisely mean that the family of morphisms $F(f_i)$, $i = 1, \dots, k$ is epimorphic. The existence of the classifying topos will then follow at once from Lemma 5.19.

As already mentioned in the previous lesson, a formula φ with free variables $a_1, \dots, a_n$ can always be seen as a formula with more free variables ... where the additional variables do not appear. This allows simplifying the language and consider that two formulæ φ and ψ have the same free variables, namely, all the free variables appearing in one of them.

First, let α and β be two coherent terms of type A with free variables $a_1, \dots, a_n$ of respective types $A_1, \dots, A_n$. In our category $\mathbb{T}$, consider the pullback

$$\begin{array}{ccc} P & \longrightarrow & A \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_A} \\ A_1 \times \cdots \times A_n & \xrightarrow[(\alpha,\beta)]{} & A \times A \end{array}$$

A left exact functor $F\colon \mathbb{T}\longrightarrow\mathcal{E}$ to a Grothendieck topos preserves all the ingredients of this diagram and thus we get the two pullbacks

$$\begin{array}{ccccc} F(P) & \longrightarrow & F(A) & \longrightarrow & \mathbf{1} \\ {\scriptstyle F(d)}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_{F(A)}} & & \big\downarrow{\scriptstyle t} \\ F(A_1) \times \cdots \times F(A_n) & \xrightarrow[\big(F(\alpha), F(\beta)\big)]{} & F(A) \times F(A) & \xrightarrow[=_A]{} & \Omega \end{array}$$

Condition 2 in Definition 1 tells us that the bottom composite is the truth table of the formula $F(\alpha) = F(\beta)$. Thus $\big[F(\alpha) = F(\beta)\big] = F(P)$ and, since $F(d)$ is a monomorphism, $F(P) = \mathsf{Im}\, F(d)$. So when $F(d)$ is an epimorphism, it is an isomorphism. We end up in this case with the family reduced to the single arrow d.

Next, consider two coherent formulæ φ and ψ with free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$. Suppose that corresponding families of morphisms

$$(f_i \colon B_i \longrightarrow A_1 \times \cdots \times A_n)_{i=1,\ldots,m} \quad (g_j \colon C_j \longrightarrow A_1 \times \cdots \times A_n)_{j=1,\ldots,k}$$

have already been associated with φ and ψ. Since by Proposition 4.9

$$[\varphi \vee \psi] = [\varphi] \cup [\psi], \quad [\varphi \wedge \psi] = [\varphi] \cap [\psi]$$

it suffices to associate the family

$$\{f_1, \ldots, f_m, g_1, \ldots, g_k\}$$

with the coherent formula $\varphi \vee \psi$ and in the case of $\varphi \wedge \psi$, the family

$$(h_{i,j})_{i=1,\ldots,m; j=1,\ldots,k}$$

obtained by pulling back each f_i along each g_j:

$$\begin{array}{ccc} H_{i,j} & \longrightarrow & B_i \\ \downarrow & \searrow^{h_{i,j}} & \downarrow{\scriptstyle f_i} \\ C_j & \xrightarrow[g_j]{} & A_1 \times \cdots \times A_n \end{array}$$

Finally let φ be a coherent formula with variables a of type A and $b_1, \ldots, b_n$ of respective types $B_1, \ldots, B_n$; write $p \colon A \times B_1 \times \cdots \times B_n \longrightarrow B_1 \times \cdots \times B_n$ for the projection. If $(f_1, \ldots, f_k)$ is the family of morphisms associated with φ, the composites $(pf_1, \ldots, pf_k)$ constitute the family associated with $\exists a\, \varphi(a, b_1, \ldots, b_n)$, just because

$$\big[\exists a\, \varphi(a, b_1, \ldots, b_n)\big] = p\big([\varphi(a, b_1, \ldots, b_n)]\big).$$

It remains to take care of the axioms. In every Heyting algebra, $1 = (a \Rightarrow b)$ iff $a \leq b$. The validity of an axiom $\models (\varphi \Rightarrow \psi)$ in a topos reduces thus to $[\varphi] \subseteq [\psi]$, that is to $[\varphi] = [\varphi] \cap [\psi]$, and further the fact that the inclusion

$$[\varphi] \cap [\psi] \rightarrowtail [\varphi]$$

is an epimorphism. One proceeds the as in the case of $\varphi \wedge \psi$ above. $\square$

Corollary 5.21 *Let $\mathcal{T}$ be a coherent theory. The category of $\mathcal{T}$-models in* Set *is equivalent to the category of points of the classifying topos $\mathcal{E}[\mathcal{T}]$.*

Sketch of proof Just by Definitions 5.13 and Theorem 5.20. $\square$

Suggestion(s) for further reading

A geometric theory $\mathcal{T}$ consists in giving a small category $\mathcal{C}$ and specifying in it a set of finite cones and a set of arbitrary cocones. The models of $\mathcal{T}$ in a Grothendieck topos $\mathcal{E}$ are the functors $\mathcal{C} \longrightarrow \mathcal{E}$ which transform the specified finite cones in limit cones and the specified cocones in colimit cocones. Allowing arbitrary small disjunctions $\bigvee_{i \in I} \varphi_i$, instead of just binary ones, yields a geometric theory. A geometric theory admits a classifying topos; moreover each Grothendieck topos is the classifying topos of a geometric theory.

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Index